GEOMETRIC INVARIANTS OF DISCRETE GROUPS

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

2026

This dissertation is dedicated to Ma. Thank you for being here.

ACKNOWLEDGEMENTS

The author would like to acknowledge his advisor Alexander Dranishnikov for his help with all projects.

The author would also like to acknowledge his colleauges Nurultan Kuanyshov for helpful discussions regarding LS-category.

The author would also like to acknowledge the UF Center of Applied Mathematic (CAM) for summer fellowship that allowed significant progress in work for the completion of this thesis.

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of the University of Florida in Partial Fulfillment of the
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May 2026

Chair: Alexander Dranishnikov
Major: Mathematics

Numerical Invariants proved to be useul in all areas of mathematics, and perhaps the most popular among them is the concept of dimension. In this thesis, my work with groups aims to explore two very different notions of dimension when applied to groups. The first notion cohomological dimension is constructed algebraically, but has strong geometric interpretation. The second notion of Lusternik-Schnirelmann Category has a more topological construction, measuring the homotopical complexity of a space. This thesis consists of three chapters corresponding to my three papers.

In the first chapter we develop the theory of cohomological dimension for group homomorphisms, extending the notion of cohomological dimension for groups. We establish a product formula $\operatorname{cd} \phi \times \phi = 2 \operatorname{cd} \phi$ for maps ϕ between geometrically finite groups. This is generalizing the same formula for groups, proved earlier by Dranishnikov. We also give counterexamples to show the classical Eilenberg-Ganea - type theorems cannot be translated directly in the context of group homomorphisms.

The second chapter concerns Right-angled Coxeter groups, which are a family of groups which come up as counterexamples (or examples) for many conjectures. About 20 years ago, Bogdan Nica conjectured that the boundary of any word-hyperbolic group admits a fixed-point-free involution. In this short chapter, we prove a variation of the conjecture, replacing word-hyperbolic groups with Right-angled Coxeter groups. This is not a complete solution to Nica's conjecture (although there is significant overlap between these two families of groups), but

hopefully the techniques used there will shed some light regarding possible approaches to the problem.

The third chapter introduces the notion of coarse Lusternik-Schnirelmann category, a large scale anaolgue of the classical Lusternik-Schnirelmann category. The classical LS category is a well-studied topological invariant, but is not a good candidate for an invariant for a group, since it is not invariant under quasi-isometries. There have been attempts to generalize the notion of LS-Category in other settings, the most relevant in this context is the *proper* LS-category, which is an invariant in the proper Homotopy Category, not under quasi-isometry. We define the *coarse* LS-category, which is a coarse invariant (not only under quasi-isometry), and show some interesting upper and lower bounds for it (by asymptotic dimension and proper LS-category respecively).

CHAPTER 1 COHOMOLOGICAL DIMENSION OF GROUP HOMOMORPHISMS

1.1 Introduction

The definition of cohomological dimension $\mathrm{cd}(G)$ of discrete group G goes back the middle of the last century. The main results of the cohomological dimension theory of groups can be found in the books (Bieri 1981) and (Brown 1982).

The cohomological dimension $\mathrm{cd}(\phi)$ of a group homomorphism $\phi : G \to H$ was defined a decade ago (Grant 2022). It was little studied ((Scott 2022),(Dranishnikov & Kuanyshov 2022)), since the concept of $\mathrm{cd}(\phi)$ is quite difficult to work with. In this paper we managed to extend some of the classic results of cohomological dimension theory from groups to group homomorphisms.

For groups together with the notion of cohomological dimension $\mathrm{cd}(G)$ there are important notions of geometric dimension $\mathrm{gd}(G)$ and homological dimension $\mathrm{hd}(G)$ (Brown 1982). The classical result of cohomological dimension theory states that $\mathrm{cd}(G) = \mathrm{gd}(G)$ whenever $\mathrm{cd}(G) \neq 2$. In the case of $\mathrm{cd}(G) = 2$ this equality is the subject of the Eilenberg-Ganea conjecture. The proof of the above equality breaks in two parts: $\mathrm{cd}(G) \geq 3$ (the Eilenberg-Ganea theorem) and $\mathrm{cd}(G) = 1$ (the Stallings-Swan theorem) (Brown 1982). For group homomorphisms there are counterexamples to both theorems. The Stallings-Swan theorem was disproved by T. Goodwillie (Grant 2022),(Dranishnikov & Kuanyshov 2022) with a group homomorphism $\phi : G \to H$ with $\mathrm{cd}(\phi) = 1$ and $\mathrm{gd}(\phi) = 2$ and the the Eilenberg-Ganea theorem was disproved in (Dranishnikov & Kuanyshov 2022) with $\mathrm{cd}(\phi) = 2$ and $\mathrm{gd}(\phi) = 3$. In this paper we show that the Eilenberg-Ganea theorem does not hold for group homomorphisms with $\mathrm{cd}(G) = n$ for any $n \geq 2$.

Generally, $\mathrm{hd}(G) \leq \mathrm{cd}(G)$ and there are groups where the inequality is strict (Brown 1982). Studying groups G with $\mathrm{hd}(G) = 1$ and $\mathrm{cd}(G) = 2$ is a quite exciting activity (see (Kropholler, Linnell & Lück 2009)). We note that for geometrically finite groups G, i.e. groups for which there is a finite classifying CW-complex $BG = K(G, 1)$, there is the equality $\mathrm{hd}(G) = \mathrm{cd}(G)$. In this paper we prove this equality for homomorphisms of geometrically finite groups.

1.1.1 Theorem. *For every homomorphism $\phi : G \to H$ of a geometrically finite group G we have* $\mathrm{hd}(\phi) = \mathrm{cd}(\phi)$.

It is known that the logarithmic law for the cohomological dimension of the product does not hold even for geometrically finite groups (Dranishnikov 1997), though the formula $\mathrm{cd}(G \times G) = 2\,\mathrm{cd}(G)$ holds (Dranishnikov 2019). In this paper we extend the latter equality to group homomorphisms between geometrically finite groups.

1.1.2 Theorem. *For every homomorphism $\phi : G \to H$ between geometrically finite groups*

$$\mathrm{cd}(\phi \times \phi) = 2\,\mathrm{cd}(\phi).$$

Among our main tools in proving the above theorems are Davis' trick with aspherical manifolds and a characterization of cohomological dimension of a group homomorphism in terms of projective resolutions which we give in this paper.

1.2 Preliminaries

1.3 Cohomology with local coefficients

Let $\pi = \pi_1(X)$, and let $\mathbb{Z}\pi$ be the group ring of π. We will refer to $\mathbb{Z}\pi$-module as to π-module. There is a bijection between local coefficients systems (locally trivial sheaves) $\mathcal{A}$ on X and π-modules where $\mathcal{A}$ corresponds to the stalk $A = \mathcal{A}_x$. A map $f : X \to Y$ and a local coefficient system $\mathcal{A}$ on Y, define a local coefficient system on X, denoted $f^*\mathcal{A}$. Given a pair of coefficient systems $\mathcal{A}$ and $\mathcal{B}$ on X, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is defined by setting $(\mathcal{A} \otimes \mathcal{B})_x = \mathcal{A}_x \otimes \mathcal{B}_x$. On the π-module level it s the tensor product of abelian groups $A \otimes B$ with the diagonal action of π.

We recall the definition of the (co)homology groups with local coefficients via π-modules (Hatcher 2002):

$$H^k(X; \mathcal{A}) \cong H^k(\mathrm{Hom}_{\mathbb{Z}\pi}(C_*(\widetilde{X}), A), \delta)$$

and

$$H_k(X; \mathcal{A}) \cong H_k(A \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}), 1 \otimes \partial)$$

where $(C_*(\widetilde{X}), \partial)$ is the chain complex of the universal cover $\widetilde{X}$ of X and A is the stalk of the local coefficient system $\mathcal{A}$, and δ is the coboundary operator. When $\mathcal{A}$ is a trivial coefficient system the above definition coincides with the usual definition in terms of the chain complex $C_*(X)$ of X. Moreover, there is the following intermediate statement.

1.3.1 Proposition. *Let $p : X' \to X$ be a normal covering and $\mathcal{A}$ local coefficient system on X such that $p^*\mathcal{A}$ is a trivial system. Then cohomology and homology groups of X with coefficients in $\mathcal{A}$ can be defined by means of the chain complex $(C_*(X'), \partial)$ as*

$$H^k(X; \mathcal{A}) \cong H^k(\mathrm{Hom}_{\mathbb{Z}\pi'}(C_*(X'), A), \delta)$$

and

$$H_k(X; \mathcal{A}) \cong H_k(A \otimes_{\mathbb{Z}\pi'} C_*(X'), 1 \otimes \partial)$$

where $\pi' = \pi_1(X)/\pi_1(X')$ is the structure group of p.

Proposition 1.3.1 follows from isomorphisms of (co)chain complexes

$$(\mathrm{Hom}_{\mathbb{Z}\pi}(C_*(\widetilde{X}), A), \delta) \cong (\mathrm{Hom}_{\mathbb{Z}\pi'}(C_*(X'), A), \delta)$$

and

$$(A \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}), 1 \otimes \partial) \cong (A \otimes_{\mathbb{Z}\pi'} C_*(X'), 1 \otimes \partial).$$

We refer to (Bredon 1997) for the definition of the cup product

$$\cup : H^i(X; \mathcal{A}) \otimes H^j(X; \mathcal{B}) \longrightarrow H^{i+j}(X; \mathcal{A} \otimes \mathcal{B})$$

and the cap product

$$\cap : H_i(X; \mathcal{A}) \otimes H^j(X; \mathcal{B}) \longrightarrow H_{i-j}(X; \mathcal{A} \otimes \mathcal{B}).$$

1.4 The Kunneth Formula and UCF

For abelian groups A and B we use notation $A * B$ for the $Tor(A, B)$.

1.4.1 Theorem (Kunneth (Bredon 1997))**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be local coefficient systems on a finite complexes X and Y with $\mathcal{A}_x * \mathcal{B}_x = 0$. Then there is a natural short exact sequence*

$$0 \to \bigoplus_p H^p(X;\mathcal{A}) \otimes H^{r-p}(Y;\mathcal{B}) \to H^r(X \times Y; \mathcal{A}\hat{\otimes}\mathcal{B}) \to \bigoplus_p H^{p+1}(X;\mathcal{A}) * H^{r-p}(Y;\mathcal{B}) \to 0.$$

When $Y = pt$ the Kunneth theorem turns into the Universal Coefficient Formula:

1.4.2 Theorem (UCF (Bredon 1997))**.** *Let $\mathcal{A}$ be a local coefficient system on a finite complexes X and let B be an abelian group with $A * B = 0$. Then there is a natural short exact sequence*

$$0 \to H^n(X;\mathcal{A}) \otimes B \to H^n(X;\mathcal{A} \otimes B) \to H^{n+1}(X;\mathcal{A}) * B \to 0.$$

1.5 Cohomological dimension of groups

By definition the cohomology of a discrete group π with coefficients in a π-module A is the cohomology of its classifying space, $H^*(\pi, A) = H^*(B\pi; \mathcal{A})$ where $\mathcal{A}$ is the corresponding local system. Also we will be using the notation $H^*(B\pi; A)$ for $H^*(\pi, A)$. It is customary to separate coefficients for group (co)homology by a coma and for the space cohomology by a semicolon.

The cohomological dimension of a group π is defined as follows

$$\mathrm{cd}(\pi) = \max\{n \mid H^n(\pi, A) \neq 0 \ \text{ for some } \ \mathbb{Z}\pi\text{-module } A\}.$$

Similarly the homological dimension is defined as

$$\mathrm{hd}(\pi) = \max\{n \mid H_n(\pi, A) \neq 0 \ \text{ for some } \ \mathbb{Z}\pi\text{-module } A\}.$$

The geometric dimension $\mathrm{gd}(\pi)$ is defined as the minimal dimension of CW-complexes serving as a classifying space $B\pi$ for π

$$\mathrm{gd}(\pi) = \min\{n \mid \exists\, B\pi \ \text{ with } \ \dim B\pi = n\}.$$

Consider the short exact sequence $0 \to \mathbf{I}_\pi \to \mathbb{Z}\pi \xrightarrow{\epsilon} \mathbb{Z} \to 0$ defined by the augmentation map ϵ. The *Berstein-Schwarz class* $\beta_\pi \in H^1(\pi, \mathbf{I}_\pi)$ of a discrete group π is the image $\delta(1)$ of

$$1 \in \mathbb{Z} = H^0(B\pi;\mathbb{Z}) \xrightarrow{\delta} H^1(B\pi;\mathbf{I}_\pi)$$

under the connecting homomorphism in the coefficients long exact sequence. Here $\mathbf{I}_\pi$ is the augmentation ideal of the group ring $\mathbb{Z}\pi$ (Berstein 1976),(Švarc 1965). The Berstein-Schwarz class is universal in the following sense.

1.5.1 Theorem (Universality (Dranishnikov & Rudyak 2009),(Švarc 1965))**.** *For any cohomology class $\alpha \in H^k(\pi, L)$ there is a homomorphism of π-modules $\mathbf{I}_\pi^k \to L$ such that the induced homomorphism for cohomology takes $\beta_\pi^k \in H^k(\pi;\mathbf{I}_\pi^k)$ to α where $\mathbf{I}_\pi^k = \mathbf{I}_\pi \otimes \cdots \otimes \mathbf{I}_\pi$ and $\beta_\pi^k = \beta_\pi \cup \cdots \cup \beta_\pi$.*

The Universality Theorem implies

1.5.2 Corollary.

$$\mathrm{cd}(\pi) = \max\{k \mid \beta_\pi^k \neq 0\}.$$

Everywhere above the background ring $\mathbb{Z}$ can be replaced by any commutative ring with unit $\mathbf{k}$ and the dimensions $\mathrm{cd}_{\mathbf{k}}(\pi)$ and $\mathrm{hd}_{\mathbf{k}}(\pi)$ can be defined accordingly. Thus, in our notations $\mathrm{cd}(\pi) = \mathrm{cd}_{\mathbb{Z}}(\pi)$. In this paper we use along with $\mathbb{Z}$ the fields $\mathbf{k} = \mathbb{Z}_p$ and $\mathbf{k} = \mathbb{Q}$.

1.5.3 Proposition. *For all $\mathbf{k}$ and π,*

$$\mathrm{cd}_{\mathbf{k}}(\pi) \leq \mathrm{cd}(\pi).$$

Proof. This follows from the fact that any $\mathbf{k}\pi$ module is canonically a $\mathbb{Z}\pi$ module. So

$$\begin{aligned}\mathrm{cd}_{\mathbf{k}}\,\pi &= \max\{n : H^n(\pi;M) \neq 0 \text{ for some } \mathbf{k}\pi\text{-module } M\}\\ &\leq \max\{n : H^n(\pi;M) \neq 0 \text{ for some } \mathbb{Z}\pi\text{-module } M\}\\ &= \mathrm{cd}\,\pi.\end{aligned}$$

□

1.6 Projective resolutions

Let π be a discrete group. A projective resolution $P_*(\pi)$ of $\mathbb{Z}$ for the group π is an exact sequence of projective π-modules

$$\cdots \to P_n(\pi) \to P_{n-1}(\pi) \to \cdots \to P_2(\pi) \to P_1(\pi) \to \mathbb{Z}\pi \xrightarrow{\epsilon} \mathbb{Z}.$$

The homology and cohomology of π with coefficients in a π-module M can be defined as homology of the chain and cochain complexes $P_*(\pi) \otimes_\pi M$ and $Hom_\pi(P_*(\pi), M)$.

The cohomological dimension $\mathrm{cd}(\pi)$ equals the shortest length of projective resolution $P_*(G)$ (Brown 1982). The same holds true for any commutative ground ring with unit $\mathbf{k}$.

1.7 Cohomological dimension of group homomorphisms

Let $\phi : G \to H$ be a group homomorphism, and let $\mathbf{k}$ be a ring. We define (see (Grant 2022))

$$\mathrm{cd}(\phi) = \max\{n \mid 0 \neq \phi^* : H^n(H, A) \to H^n(G, A) \text{ for some } \mathbb{Z}H\text{-module } A\}.$$

Similarly the homological dimension is defined as

$$\mathrm{hd}(\phi) = \max\{n \mid 0 \neq \phi^* : H_n(G, A) \to H_n(H, A) \text{ for some } \mathbb{Z}H\text{-module } A\}.$$

As before, everywhere the background ring Z can be replaced with a commutative ring $\mathbf{k}$ with identity, to get $\mathrm{cd}_{\mathbf{k}}(\phi)$ and $\mathrm{hd}_{\mathbf{k}}(\phi)$. Note that for the identity homomorphism $1_G : G \to G$ we recover the definition of $\mathrm{cd}(G)$ and $\mathrm{hd}(G)$.

Note that the proof of 1.5.3 translates directly to the proof of the following:

1.7.1 Proposition. *For any commutative ring* $\mathbf{k}$ *with identity, and group homomorphism* $\phi : G \to H$ *we have*

$$\mathrm{cd}_{\mathbf{k}}(\phi) \leq \mathrm{cd}(\phi).$$

Any group homomorphism $\phi : G \to H$ can be realized by a cellular map $\bar{\phi} : BG \to BH$ as $\phi = \bar{\phi}_* : \pi_1(BG) \to \pi_1(BH)$. The geometric dimension $\mathrm{gd}(\phi)$ of ϕ is defined as the minimal n such that $\bar{\phi}$ can be deformed to the n-skeleton $BH^{(n)}$:

$$\mathrm{gd}(\phi) = \min\{n \mid \bar{\phi} \sim f : BG \to BH^{(n)}\}.$$

The Lusternik-Schnirelmann category $\mathrm{cat}(f)$ of a map $f : X \to Y$ is defined as the minimal number k such that X can be covered by $k + 1$ open sets $U_0, U_1, \ldots, U_k$ with null-homotopic restrictions $f|_{U_i} : U_i \to Y$ for all i. This definition can be extended to group homomorphisms $\phi : G \to H$ as $\mathrm{cat}(\bar{\phi})$ (Scott 2022). The following is well-known (see for example Proposition 4.3 in (Dranishnikov 2015)).

1.7.2 Proposition. *For a group homomorphism $\phi : G \to H$ with its realizing map the $\bar{\phi} : BG \to BH$ the following are equivalent:*

(1) $\mathrm{gd}(\phi) \leq n$*;*

(2) $\mathrm{cat}(\bar{\phi}) \leq n$*.*

We note that in the case of the identity homomorphism $1_G : G \to G$ our definition of the geometric dimension gives the category $\mathrm{cat}(G)$ which by the Eilenberg-Ganea theorem (Eilenberg & Ganea 1957) coincides with $\mathrm{cd}(G)$. The later equals $\mathrm{gd}(G)$ with a potential exception when $\mathrm{cd}(G) = 2$.

We recall that any group homomorphism $\varphi : G \to H$ defines a chain map between projective resolutions $\varphi_* : P_*(G) \to P_*(H)$ which induces homomorphism of (co)homology groups (Brown 1982). A chain homotopy between two chain maps $\varphi_*, \psi_* : P_*(G) \to P_*(H)$ is a sequence of homomorphisms $D_* : P_*(G) \to P_{*+1}(H)$ such that $\partial D_* + D_*\partial = \varphi_* - \psi_*$. Two chain homotopic homomorphisms induce the same homomorphisms for homology and cohomology (Hatcher 2002).

1.7.3 Theorem. *Let $\varphi : G \to H$ be a group homomorphism with* $\mathrm{cd}(\varphi) = n$ *and let $\varphi_* : (P_*(G), \partial_*) \to (P_*(H), \partial'_*)$ be the chain map between projective resolutions of $\mathbb{Z}$ for G and*

H induced by φ. Then φ_* is chain homotopic to $\psi_* : P_*(G) \to P_*(H)$ with $\psi_k = 0$ for $k > n$.

Proof. Let $d'_{n+1} : P_{n+1}(H) \to K_n = \mathrm{Im}(\partial'_{n+1})$ be the range restriction for ∂'_{n+1}. We have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+2}(G) \arrow[r, "\partial_{n+2}"] \arrow[d, "\varphi_{n+2}"] & P_{n+1}(G) \arrow[r, "\partial_{n+1}"] \arrow[d, "\varphi_{n+1}"] & P_n(G) \arrow[r] \arrow[d, "\varphi_n"] & \cdots \\
\cdots \arrow[r] & P_{n+2}(H) \arrow[r, "\partial'_{n+2}"] \arrow[dr, "d'_{n+2}"] & P_{n+1}(H) \arrow[r, "\partial'_{n+1}"] \arrow[dr, "d'_{n+1}"] & P_n(H) \arrow[r] & \cdots \\
\cdots & K_{n+2} \arrow[u, "\subset"] & K_{n+1} \arrow[u, "\subset"] & K_n \arrow[u, "\subset"] & \cdots
\end{tikzcd}$$

Since $\partial'_{n+2} d'_{n+1} = 0$, we obtain that d'_{n+1} is a cocycle in the cochain complex $\mathrm{Hom}_H(P_*(H), K_n)$. Therefore, it represents an element $[d'_{n+1}] \in H^{n+1}(H, K_n)$. Since $\mathrm{cd}(\varphi) = n$, we get that $\varphi^*[d'_{n+1}] = 0$. That is,

$$d_n := d'_{n+1}\varphi_{n+1} : P_{n+1}(G) \to K_n$$

is a coboundary in the cochain complex $\mathrm{Hom}_G(P_*(G), K_n)$. So there exists a map $h_n : P_n(G) \to K_n$ such that $h_n \partial_{n+1} = d_n$. But since $P_n(G)$ is a projective G-module and d'_{n+1} is surjective, the map h_n can be lifted to a G-homomorphism $D_n : P_n(G) \to P_{n+1}(H)$, such that $d'_{n+1} D_n = h_n$. Note that the D_n as constructed has the property

$$\begin{aligned}
\partial'_{n+1}(\varphi_{n+1} - D_n\partial_{n+1}) &= \partial'_{n+1}\varphi_{n+1} - \partial'_{n+1}D_n\partial_{n+1} \\
&= \partial'_{n+1}\varphi_{n+1} - h_n\partial_{n+1} = 0.
\end{aligned}$$

Now assume that we have constructed D_{n+i-1} such that

$$\partial'_{n+i}(\varphi_{n+i} - D_{n+i-1}\partial_{n+i}) = 0.$$

Since the bottom row is exact,

$$\mathrm{Im}(\varphi_{n+i} - D_{n+i-1}\partial_{n+i}) \subset K_{n+i} = \mathrm{Im}(\partial'_{n+i+1}).$$

Since $P_{n+i}(G)$ is a projective G-module the G-homomorphism

$$\varphi_{n+i} - D_{n+i-1}\partial_{n+i} : P_{n+i}(G) \to K_{n+i}$$

can be lifted to a G-homomorphism $D_{n+i} : P_{n+i}(G) \to P_{n+i+1}(H)$. Since

$$\partial'_{n+i+1} D_{n+i} = \varphi_{n+i} - D_{n+i-1}\partial_{n+i} \tag{1-1}$$

we obtain

$$\begin{aligned}
\partial'_{n+i+1}(\varphi_{n+i+1} - D_{n+i}\partial_{n+i+1}) &= \partial'_{n+i+1}\varphi_{n+i+1} - \partial'_{n+i+1} D_{n+i}\partial_{n+i+1} \\
&= \partial'_{n+i+1}\varphi_{n+i+1} - (\varphi_{n+i} - D_{n+i-1}\partial_{n+i})\partial_{n+i+1} \\
&= \partial'_{n+i+1}\varphi_{n+i+1} - \varphi_{n+i}\partial_{n+i+1} - D_{n+i-1}\partial_{n+i}\partial_{n+i+1} \\
&= \partial'_{n+i+1}\varphi_{n+i+1} - \varphi_{n+i}\partial_{n+i+1} \\
&= 0
\end{aligned}$$

Here we are using the facts that $\partial\partial = 0$ and φ_* is a chain map.

Therefore we can inductively construct D_k's for all $k \geq n$. Define $D_k : P_k(G) \to P_{k+1}(H)$ to be the zero homomorphisms for all $k < n$. Therefore we have constructed a chain homotopy $D_* : P_*(G) \to P_{*+1}(H)$ between φ_* and a chain map ψ_* defined by

$$\psi_k = \varphi_k - \partial'_{k+1} D_k - D_{k-1}\partial_k.$$

In view of (1-1) we obtain that $\psi_k = 0$ for all $k > n$. □

1.8 Reduction to a map of aspherical manifold

1.8.1 Proposition. *Let F be any of the numerical invariants* $\mathrm{cd}_{\mathbf{k}}$, $\mathrm{hd}_{\mathbf{k}}$, *or* gd. *Then for homomorphisms* $A \xrightarrow{g} G \xrightarrow{f} H$

$$F(f \circ g) \leq \min\{F(f), F(g)\}.$$

Proof. The statement is obvious for the cohomological and homological dimensions. Clearly, $\mathrm{gd}(f \circ g) \leq \mathrm{gd}(f)$. We may assume that the map $\bar{f} : BH \to BA$ realizing f is cellular. Then the inequality $\mathrm{gd}(f \circ g) \leq \mathrm{gd}(g)$ follows. □

1.8.2 Corollary. *Suppose that $r : BA \to BG$ is a retraction. Then for any $f : BG \to BH$ and any above choice of the numerical invariant F, $F(f \circ r) = F(f)$.*

Proof. Let $i : BG \to BA$ be the inclusion for which $r \circ i = 1$. Then

$$F(f \circ r) \leq F(f) = F((f \circ r) \circ i) \leq F(f \circ r).$$

□

1.8.3 Corollary. *For every homomorphism $\phi : G \to H$ of a geometrically finite group G there is a closed aspherical orientable manifolds M containing BG as a retract and a map $g : M \to BH$ such that $\phi = (g|_{BG})_* : \pi_1(BG) \to \pi_1(BH)$ and $F(\phi) = F(g)$ for the above choice of F.*

Proof. By Davis' trick (Davis 1983) for every group G with finite complex BG there is a closed aspherical orientable manifold containing BG as a retract. □

1.8.4 Theorem. *Let homomorphism $\phi : G \to H$ factor as $\phi = j \circ \phi'$ where ϕ' is surjective and j is injective. Then for any choice F of the numerical invariants* $\mathrm{cd}_{\mathbf{k}}$, $\mathrm{hd}_{\mathbf{k}}$, *or* gd *we have* $F(\phi) = F(\phi')$.

Proof. In view of the equality $\mathrm{gd}(\phi) = \mathrm{cat}(\phi)$, Theorem 3.2 and Theorem 3.3 in (Dranishnikov & Kuanyshov 2022), the statement is proven when $F = \mathrm{gd}$ ands $F = \mathrm{cd}$.

By Proposition 1.8.1 $\mathrm{hd}(\phi) \leq \mathrm{hd}(\phi')$. Let $\mathrm{hd}(\phi') = k$. We show that $\mathrm{hd}(\phi) \geq k$. Let $\pi = \phi'(G)$ and let $\phi'_* : H_k(G, M) \to H_k(\pi, M)$ be a nonzero homomorphism for some π-module M. Let i denote the inclusion $M \cong 1 \otimes M \xrightarrow{\subset} \mathbb{Z}H \otimes_\pi M = \mathrm{Ind}_\pi^H M$ of M into the induced H-module. We consider the following commutative diagram generated by i, the inclusion

$j : \pi \to H$, and ϕ'

$$\begin{tikzcd}
H_k(G, M) \arrow[r, "i_*"] \arrow[d, "\phi'_*"'] & H_k(G, \mathrm{Ind}_\pi^H M) \arrow[d, "\phi'_*"'] & \\
H_k(\pi, M) \arrow[r, "i_*"] & H_k(\pi, \mathrm{Ind}_\pi^H M) \arrow[r, "j_*"] & H_k(H, \mathrm{Ind}_\pi^H M).
\end{tikzcd}$$

The bottom composition $j_* i_*$ is the Shapiro Lemma isomorphism (Brown 1982). Therefore, $j_* i_* \phi'_* \neq 0$. Thus, the composition

$$\phi_* = j_* \phi'_* : H_k(G, \mathrm{Ind}_\pi^H M) \to H_k(H, \mathrm{Ind}_\pi^H M)$$

is not zero. Hence $\mathrm{hd}(\phi) \geq k$.

The same proof works for any ground ring $\mathbf{k}$. □

1.8.5 Corollary. *Let homomorphism $\phi : G \to H$ factor as $\phi = j \circ \phi'$ where j is injective. Then for any choice F of the numerical invariants* $\mathrm{cd}_{\mathbf{k}}$, $\mathrm{hd}_{\mathbf{k}}$, *or* gd *we have $F(\phi) = F(\phi')$.*

Proof. Let $\phi' : G \to H'$, $j : H' \to H$, and let $\phi'' : G \to \mathrm{Im}\,\phi$ denote the range restriction of ϕ. Let $i : \mathrm{Im}(G) \to H'$ be the inclusion. Then by Theorem 1.8.4 applied to $\phi = (ij)\phi''$ and $\phi' = i\phi''$,

$$F(\phi) = F((ij)\phi'') = F(\phi'') = F(i\phi'') = F(\phi').$$

□

1.8.6 Corollary. *If the group H in Corollary 1.8.3 is geometrically finite, then there is a closed orientable aspherical manifold N containing BH and a map $f : M \to N$ such that $f(BG) \subset BH$, $\phi = (f|_{BG})_* : \pi_1(BG) \to \pi_1(BH)$ and $F(\phi) = F(g)$ for the above choice of F.*

Proof. If H is geometrically finite we consider the embedding $i_H : BH \to N$ of BH into a closed aspherical orientable manifold from the Davis' trick. Let $f = i_H \circ g : M \to N$. By Theorem 1.8.4, $F(f) = F(g)$. □

We call a group homomorphism $\phi : G \to H$ a *subhomomorphism* of a group homomorphism $\phi' : G' \to H'$ if there are inclusions as a subgroup $i : G \to G'$ and $j : H \to H'$ such that $\phi = j\phi' i$.

1.8.7 Proposition. *Let $\phi : G \to H$ be a subhomomorphism of $\phi' : G' \to H'$. Then $F(\phi) \leq F(\phi')$ for the above choice of F.*

Proof. By Corollary 1.8.5, the equality $j\phi = \phi' i$, and Proposition 1.8.1 we obtain

$$F(\phi) = F(j\phi) = F(\phi' i) \leq F(\phi').$$

□

1.8.8 Proposition. *Suppose that a local coefficient system $\mathcal{A}$ on a CW complex X pulls back to a trivial system $(p')^*\mathcal{A} = X' \times A$ on X' by a normal covering map $p' : X' \to X$. Then every homology class $a \in H_k(X; \mathcal{A})$ can be detected by a cohomology class $\gamma \in H^k(X; \mathcal{B})$ with some local coefficient system $\mathcal{B}$ with $(p')^*\mathcal{B}$ trivial on X', that is $0 \neq a \cap \gamma \in H_0(X, \mathcal{A} \otimes \mathcal{B})$.*

Proof. Consider the chain complex

$$\ldots \longrightarrow C_{k+1}(X') \xrightarrow{\partial'_{k+1}} C_k(X') \xrightarrow{\partial'_k} C_{k-1}(X') \longrightarrow \ldots \ .$$

We set $B' := C_k(X')/\operatorname{Im} \partial'_{k+1}$. Let $\mathcal{B}$ be the corresponding local system on X. Let $p : \widetilde{X} \to X$ be the universal covering and let $p = p' \circ q$. The map q defines the following commutative diagram:

$$\begin{array}{ccccccc}
C_{k+1}(\widetilde{X}) & \xrightarrow{\partial_{k+1}} & C_k(\widetilde{X}) & \xrightarrow{\widetilde{f}} & \widetilde{B} & \longrightarrow & 0 \\
{\scriptstyle q_*}\downarrow & & {\scriptstyle q_*}\downarrow & & {\scriptstyle q_*}\downarrow & & \\
C_{k+1}(X') & \xrightarrow{\partial'_{k+1}} & C_k(X') & \xrightarrow{f'} & B' & \longrightarrow & 0
\end{array}$$

where $\widetilde{B} := C_k(\widetilde{X})/\operatorname{Im} \partial_{k+1}$. Since the chain complex $C_*(\widetilde{X})$ is exact, $\widetilde{B} = C_k(\widetilde{X})/\operatorname{Ker} \partial_k = \operatorname{Im} \partial_k$.

Then $\widetilde{f}$ can be identified as the range restriction of ∂_k. We define $f = f' \circ q_* : C_k(\widetilde{X}) \to B'$. Since

$$\delta f(x) = f(\partial_{k+1}(x)) = q_*((\widetilde{f} \circ \partial_{k+1})(x)) = q_*(\partial_k \circ \partial_{k+1})(x)) = 0,$$

the homomorphism f can be regarded as a k-cocycle with values in B'. Let $\gamma := [f] \in H^k(X; \mathcal{B})$ be the cohomology class of f.

Now we prove that $a \cap \gamma \neq 0$. In view of Proposition 1.3.1 we can represent the class a by a cycle $z \in A \otimes C_k(\widetilde{X})$ where A be the stalk of $\mathcal{A}$. Since $z \notin \mathrm{Im}(1 \otimes \partial_{k+1})$, we conclude that $(1 \otimes \widetilde{f})(z) \neq 0 \in A \otimes \widetilde{B}$.

The tensor product of the above diagram with A over the group ring $\mathbb{Z}\pi$ where $\pi = \pi_1(X)$ gives the following commutative diagram with exact rows

$$\begin{tikzcd}
A \otimes_\pi C_{k+1}(\widetilde{X}) \arrow[r, "1\otimes\partial_{k+1}"] \arrow[d, "1\otimes q_*"', "\cong"] & A \otimes_\pi C_k(\widetilde{X}) \arrow[r, "1\otimes\widetilde{f}"] \arrow[d, "1\otimes q_*"', "\cong"] & A \otimes_\pi \widetilde{B} \arrow[r] \arrow[d, "1\otimes q_*"', "\cong"] & 0 \\
A \otimes_\pi C_{k+1}(X') \arrow[r, "1\otimes\partial'_{k+1}"] & A \otimes_\pi C_k(X') \arrow[r, "1\otimes f'"] & A \otimes_\pi B' \arrow[r] & 0
\end{tikzcd}$$

Since the action of the group $\pi' = \pi_1(X') \subset \pi$ on A is trivial, the left two vertical arrows in the diagram are isomorphisms. By the Five Lemma the right vertical arrow is an isomorphism as well. Then

$$(1 \otimes f)(z) = (1 \otimes f')(1 \otimes q_*)(z) = (1 \otimes q_*)(1 \otimes \widetilde{f})(z) \neq 0 \in A \otimes_\pi B' = H_0(X; \mathcal{A} \otimes \mathcal{B}).$$

Thus, for the cohomology class γ of f we have $a \cap \gamma \neq 0$. $\square$

1.8.9 Theorem. *For any choice of commutative ring with unit* $\mathbf{k}$, *for every homomorphism* $\phi : G \to H$ *of geometrically finite group* G,

$$\mathrm{hd}_{\mathbf{k}}(\phi) = \mathrm{cd}_{\mathbf{k}}(\phi).$$

Proof. Since the proof in the case of general ground ring $\mathbf{k}$ is the same as for $\mathbb{Z}$, one may assume that everything below is performed over the ring of integers. In view of Theorem 1.8.4 it suffices to prove this theorem for the case when ϕ is an epimorphism. Let $r : M \to BG$ be a retraction of a closed orientable aspherical manifold as in Corollary 1.8.3. We define $g = \bar{\phi} r : M \to BH$ where $\bar{\phi} : BG \to BH$ is a map realizing ϕ on the fundamental groups. In view of Corollary 1.8.3, it suffices to show that $\mathrm{hd}(g) = \mathrm{cd}(g)$. Let $M' = g^*EH$ be the pull-back of the universal covering of BH. Suppose that $\mathrm{cd}(g) = k$. Then $g^*(\beta_H^k) \neq 0$. By the Poincare Duality with local coefficients, $a = [M] \cap g^*(\beta_H^k) \neq 0$. We note that the local coefficient system $\mathcal{A}$ on M for the class $g^*(\beta_H^k)$ pulls back by g from the system on BH defined by the H-module $\mathbf{I}_H^k$. The same system serves as coefficients for the dual homology class $a \in H_{|a|}(M; \mathcal{A})$. Since EH is contractible, the system $\mathcal{A}$ pulls back to a trivial system on M' by the covering $M' \to M$. By Proposition 1.8.8 there is a cohomology class $\gamma \in H_{|a|}(M; \mathcal{B})$ with a local system $\mathcal{B}$ on M, that pulls back to a trivial system on M', detecting the homology class a. Thus,

$$([M] \cap \gamma) \cap g^*\beta_H^k \in H_0(M; \mathcal{A} \otimes \mathcal{B}) \neq 0.$$

Since all coefficients are coming from BH, the induced homomorphism g_* for homology is well-defined. Since g is an epimorphism of the fundamental groups, g_* is an isomorphism in dimension 0. Thus, we obtain

$$0 \neq g_*(([M] \cap \gamma) \cap g^*\beta_H^k) = g_*([M] \cap \gamma) \cap \beta_H^k.$$

Hence, $g_*([M] \cap \gamma) \neq 0$. Note that $[M] \cap \gamma \in H_k(M; \mathcal{B})$. Thus, $\mathrm{hd}(g) \geq k$.

By Theorem 1.7.3 the chain map $g_* : C_*(\widetilde{M}) \to C_*(EH)$ is chain homotopic to a chain map q_* with $q_i = 0$ for $i > k$. Then for any H-module $1_M \otimes q_i = 0$ for $i > k$. Therefore, the induced homomorphism of homology is 0 in dimension greater than k. Thus, $\mathrm{hd}(g) \leq k$. □

We note that the proof of the inequality $\mathrm{hd}(\phi) \leq \mathrm{cd}(\phi)$ works for any group homomorphisms.

1.9 Cohomological dimension with respect to a field

For a discrete group π we consider the chain complex $(C_*(E\pi), \partial_*)$. We use notations $C_k = C_k(\pi) = C_k(E\pi)$ for the group of k-chains and $B_k = B_k(\pi) = \operatorname{Im} \partial_k$ for the group of $(k-1)$-boundaries. Since the chain complex $(C_*(E\pi), \partial_*)$ is acyclic, there are short exact sequences

$$0 \to B_{k+1} \to C_k \to B_k \to 0. \qquad (1\text{-}2)$$

1.9.1 Proposition. *Let $\Lambda = \pi_1(M)$ be the fundamental group of a closed orientable aspherical n-manifold M. Then $H^k(\Lambda, B_k(\Lambda)) = \mathbb{Z}$ for $k \leq n$.*

Proof. We may assume that M has one n-dimensional cell and, hence, $C_n = \mathbb{Z}\Lambda$. Note that $B_n \cong C_n$ and hence $H^n(\Lambda, B_n) = H^n(M; \mathbb{Z}\Lambda) = H^n_c(\widetilde{M}; \mathbb{Z}) = \mathbb{Z}$.

By results of McMillan (Jr 1961), McMillan-Zeeman (McMillan & Zeeman 1962), and Stallings (Stallings 1962) proven almost simultaneously in early 60s, we obtain $\widetilde{M} \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$ for some (all) $m \geq 1$. Hence the reduced suspension $\Sigma^m(\alpha\widetilde{M})$ of the one point compactification $\alpha\widetilde{M}$ of the universal covering $\widetilde{M}$ of M is homeomorphic to S^{n+m}. Therefore,

$$H^k(\Lambda, \mathbb{Z}\Lambda) = H^k_c(\widetilde{M}; \mathbb{Z}) = \bar{H}^k(\alpha\widetilde{M}) = 0$$

for $k < n$. Since the chai groups $C_i(\pi) = \oplus^{m_i}\mathbb{Z}\Lambda$ are finitely generated free Λ-modules, we obtain from the coefficient exact sequence (1-2), $H^{k-1}(\Lambda, B_{k-1}) = H^k(\Lambda, B_k)$ for $k < n$. Then the equality $H^0(\Lambda, B_0) = \mathbb{Z}$ implies the equality $H^k(\Lambda, B_k) = \mathbb{Z}$ for $k < n$. $\square$

The following Proposition is a refinement of the inequality $\operatorname{hd}(\pi) \leq \operatorname{cd}(\phi)$.

1.9.2 Proposition. *Let $\phi : \Gamma \to \pi$ be a group homomorphism. Assume that $\phi_* : H_k(\Gamma, A) \to H_k(\pi, A)$ is a nontrivial homomorphism for some π-module A. Then $\phi^* : H^k(B\pi; B_k(\pi)) \to H^k(B\Gamma; B_k(\pi))$ is a nontrivial homomorphism.*

Proof. Clearly, the natural epimorphism $f : C_k(\pi) \to B_k(\pi) = B_k$ is a cocycle. Let $\alpha = [f] \in H^k(B\pi; B_k)$. We show that $\phi^*(\alpha) \neq 0$. In fact we show that $a \cap \phi^*(\alpha) \neq 0$ for any

$a \in H_k(\Gamma, A)$ with $\phi_*(a) \neq 0$. Let $z \in A \otimes_\pi C_k(\Gamma)$ be a cycle representing a. Then $z' = (1 \otimes \phi_*)(z) \in A \otimes_\pi C_k(\pi)$ is a cycle representing $\phi_*(a) \neq 0$. Therefore, $z' \notin \mathrm{Im}(1 \otimes \partial_{k+1})$. Hence,

$$0 \neq (1 \otimes f)(z') \in A \otimes_\pi B_k = H_0(\pi, A \otimes B_k).$$

Thus, $\phi_*(a) \cap \alpha \neq 0$. Note that $\phi_*(a) \cap \alpha = \phi_*(a \cap \phi^*(\alpha))$. Therefore, $a \cap \phi^*(\alpha) \neq 0$ and hence $\phi^*(\alpha) \neq 0$. □

1.9.3 Theorem. *For any homomorphism $\varphi : \Gamma \to \pi$ between geometrically finite groups, there is a field $\mathbf{k}$ such that* $\mathrm{cd}(\varphi) = \mathrm{cd}_{\mathbf{k}}(\varphi)$.

Proof. By Corollary 1.8.6 we may assume that φ is realized by a map of closed aspherical orientable manifolds. Let $\mathrm{cd}(\varphi) = k$. Then by Theorem 1.8.9, $\mathrm{hd}(\varphi) = k$. By Proposition 1.9.2

$$\varphi^* : H^k(B\pi; B_k(\pi)) \to H^k(B\Gamma; B_k(\pi))$$

is a nontrivial homomorphism. By virtue of Proposition 1.9.1 the image

$$A = \varphi^*(H^k(\pi, B_k(\pi))) \neq 0$$

is a cyclic group.

Case 1: The group $A = \mathbb{Z}$. Then $A \otimes \mathbb{Q} \neq 0$, and considering the $\mathbb{Q}\Gamma$-module $B_k(\pi) \otimes \mathbb{Q}$, we see that $\mathrm{cd}_{\mathbb{Q}}(\varphi) \geq k$. Using Proposition 1.7.1 we get $\mathrm{cd}_{\mathbb{Q}}(\varphi) = k$, and we are done.

Case 2: The group A is of finite order m. Let prime p be a divisor of m. Then the homomorphism

$$H^k(\pi, B_k(\pi)) \otimes \mathbb{Z}_p \to A \otimes \mathbb{Z}_p$$

is a nontrivial epimorphism. Since the group $B_k(\pi) \subset C_{k-1}(\pi)$ is torsion free, we have $B_k(\pi) * \mathbb{Z}_p = 0$. Therefore, the Universal Coefficient Formula (UCF) holds for $B\pi$. By the UCF there is a commutative diagram of exact sequences

$$
\begin{array}{ccccc}
0 & \longrightarrow & H^k(B\pi; B_k(\pi)\otimes\mathbb{Z}_p & \longrightarrow & H^n(B\pi; B_k(\pi)\otimes\mathbb{Z}_p) \\
 & & \Big\downarrow{\scriptstyle \neq 0} & & \Big\downarrow{\scriptstyle \varphi_p^*} \\
0 & \longrightarrow & H^k(B\Gamma; B_k(\pi)\otimes\mathbb{Z}_p & \longrightarrow & H^n(B\Gamma; B_k(\pi)\otimes\mathbb{Z}_p)
\end{array}
$$

which implies that the homomorphism

$$\varphi_p^* : H^n(B\pi; B_k(\pi)\otimes\mathbb{Z}_p) \to H^n(B\Gamma; B_k(\pi)\otimes\mathbb{Z}_p)$$

is nonzero.

Therefore, taking $M = B_k(\pi)\otimes\mathbb{Z}_p$ as our $\mathbb{Z}_p\pi$-Module, we see that the map $H^k(\pi, M) \to H^k(\Gamma, M)$ induced by φ is nonzero. Therefore, $\mathrm{cd}_{\mathbb{Z}_p}(\varphi) \geq k$. Again, by Proposition 1.7.1 we conclude that $\mathrm{cd}_{\mathbb{Z}_p}(\varphi) = \mathrm{cd}(\varphi)$. □

1.10 Cohomological dimension of the product

1.10.1 Lemma. *For any two homomorphisms $\phi : \Gamma \to \pi$ and $\phi' : \Gamma' \to \pi'$ and any commutative ring R there is the inequality* $\mathrm{cd}_R(\phi\times\phi') \leq \mathrm{cd}_R\,\phi + \mathrm{cd}_R\,\phi'$.

Proof. Consider projective resolutions $P_*(\Gamma)$, $P_*(\Gamma')$, $P_*(\pi)$, and $P_*(\pi')$ of R for $R\Gamma$, $R\Gamma'$, $R\pi$, and $R\pi'$ modules respectfully. Let $\phi_* : P_*(\Gamma) \to P_*(\pi)$ and $\phi'_* : P_*(\Gamma') \to P_*(\pi')$ be chain maps between projective resolutions generated by ϕ and ϕ'. Let $\mathrm{cd}_R(\phi) = m$ and $\mathrm{cd}_R(\phi') = n$. By Theorem 1.7.3 ϕ_* is chain homotopic to a chain map ψ_* with $\psi_i(P_i(\Gamma)) = 0$ for $i > m$. Similarly, ϕ'_* is chain homotopic to a chain map ψ'_* with $\psi_j(P_j(\Gamma')) = 0$ for $j > n$. Then the chain map

$$(\phi\times\phi')_* : (P(\Gamma)\otimes_R P(\Gamma'))_* \to (P(\pi)\otimes_R P(\pi'))_*$$

is chain homotopic to $(\psi\times\psi')_*$. Note that $(\psi\times\psi')_k = 0$ for $k > m+n$. Therefore, the induced homomorphism

$$(\psi\times\psi')^* = (\phi\times\phi')^* : H^k(\pi\times\pi', M) \to H^k(\Gamma\times\Gamma', M)$$

is 0 for $k > m + n$ for any $R(\pi \times \pi')$-module M. Hence $\mathrm{cd}_R(\phi \times \phi') \leq m + n$. □

1.10.2 Lemma. *For any two homomorphisms $\varphi : \Gamma \to \pi$ and $\psi : \Gamma' \to \pi'$ between geometrically finite groups and for any field $\mathbf{k}$ there is the inequality*

$$\mathrm{cd}_{\mathbf{k}}(\varphi \times \psi) \geq \mathrm{cd}_{\mathbf{k}}(\varphi) + \mathrm{cd}_{\mathbf{k}}(\psi).$$

Proof. We recall that there is the Kunneth Formula for the sheaf cohomology of compact spaces (Bredon 1997) which in the case of a field degenerates into a natural isomorphism. Thus, for any geometrically finite group G and any $\mathbf{k}G$-module M for a field $\mathbf{k}$ there is an isomorphism:

$$\bigoplus_{p+q=r} H^p(G; M) \otimes_{\mathbf{k}} H^q(G; M) \to H^r(G \times G; M \otimes_{\mathbf{k}} M').$$

Let $m = \mathrm{cd}_{\mathbf{k}}(\varphi)$ and Let $n = \mathrm{cd}_{\mathbf{k}}(\psi)$. Let M to be a $\mathbf{k}\pi$-module for which
$\varphi^* : H^n(\pi; M) \to H^n(\Gamma; M)$ is nonzero and let M' to be a $\mathbf{k}\pi'$-module for which
$\psi^* : H^n(\pi'; M') \to H^n(\Gamma'; M')$ is nonzero. We obtain the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \bigoplus_{p+q=r} H^p(\pi, M) \otimes_{\mathbf{k}} H^q(\pi', M') & \longrightarrow & H^r(\pi \times \pi', M \otimes_{\mathbf{k}} M') & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \bigoplus \varphi^* \otimes \psi^*} & & \Big\downarrow {\scriptstyle (\varphi \times \psi)^*} & & \\
0 & \longrightarrow & \bigoplus_{p+q=r} H^p(\Gamma; M) \otimes_{\mathbf{k}} H^q(\Gamma'; M) & \longrightarrow & H^r(\Gamma' \times \Gamma; M \otimes_{\mathbf{k}} M') & \longrightarrow & 0
\end{array}$$

For $r = m + n$ on the left hand side the summand with $p = m$ and $q = n$ gives us a nonzero homomorphism of vector spaces over $\mathbf{k}$. By commutativity, we get that the right vertical map is also nonzero in dimension $m + n$. Therefore, $\mathrm{cd}_{\mathbf{k}}(\varphi \times \psi) \geq m + n$. □

1.10.3 Corollary. *For any two homomorphisms $\varphi : \Gamma \to \pi$ and $\psi : \Gamma' \to \pi'$ between geometrically finite groups and for any field $\mathbf{k}$ there is the equality*

$$\mathrm{cd}_{\mathbf{k}}(\varphi \times \psi) = \mathrm{cd}_{\mathbf{k}}(\varphi) + \mathrm{cd}_{\mathbf{k}}(\psi).$$

Proof. Apply Lemma 1.10.1 and Lemma 1.10.2. □

1.10.4 Corollary. *For any homomorphism $\varphi : \Gamma \to \pi$ between two geometrically finite groups there is the inequality* $\mathrm{cd}(\varphi \times \varphi) \geq 2\,\mathrm{cd}(\varphi)$.

Proof. By Theorem 1.9.3 there is a field $\mathbf{k}$ such that $\mathrm{cd}(\varphi) = \mathrm{cd}_{\mathbf{k}}(\varphi)$. Then in view of Proposition 1.7.1

$$\mathrm{cd}(\varphi \times \varphi) \geq \mathrm{cd}_{\mathbf{k}}(\varphi \times \varphi) = 2\,\mathrm{cd}_{\mathbf{k}}(\varphi) = 2\,\mathrm{cd}(\varphi).$$

□

1.10.5 Theorem. *For any homomorphism $\varphi : \Gamma \to \pi$ between two geometrically finite groups there is the equality* $\mathrm{cd}(\varphi \times \varphi) = 2\,\mathrm{cd}(\varphi)$.

Proof. Apply Lemma 1.10.1 with $R = \mathbb{Z}$ and Corollary 1.10.4. □

1.11 On the analog of the Eilenberg-Ganea theorem for homomorphisms

The Eilenberg-Ganea equality (Brown 1982) $\mathrm{cd}(\pi) = \mathrm{gd}(\pi)$ holds true whenever $\mathrm{cd}(\pi) \geq 3$. The Eilenberg-Ganea conjecture extends this equality to the case of $\mathrm{cd}(\pi) = 2$. A potential counter-example should have $\mathrm{gd}(\pi) = 3$. In (Dranishnikov & Kuanyshov 2022) a map $f : W^4 \to T^3$ of an aspherical 4-manifold W^4 onto a 3-torus is constructed satisfying $\mathrm{cd}(f) = 2$ and $\mathrm{gd}(f) = 3$. Below we show the fact that the numbers 2 and 3 in this example are the same as in the Eilenberg-Ganea conjecture is rather coincidental.

We recall the notation

$$\pi_s^k(X) = \varinjlim [\Sigma^n X, S^{n+k}]$$

for the stable k-cohomotopy group of X.

1.11.1 Proposition. *Suppose that the map $f : W \to T$ induces a nontrivial homomorphism $f^* : \pi_s^k(T) \to \pi_s^k(W)$ and satisfies* $\mathrm{cd}(f) < k$. *Then the map $f \times 1 : W \times S^1 \to T \times S^1$ induces a nontrivial homomorphism*

$$(f \times 1)^* : \pi_s^{k+1}(T \times S^1) \to \pi_s^k(W \times S^1)$$

and satisfies the inequality $\mathrm{cd}(f \times 1) < k + 1$.

Proof. Let $S^1 = I_+ \cup I_-$ with $I_+ \cap I_- = S^0 = \{x_0, -x_0\}$. We use the relative Mayer-Vietoris sequence for π_s^* to show the isomorphism

$$\pi_s^{i-1}(Y) = \pi_s^{i-1}(Y \times -x_0) = \pi_s^{i-1}(Y \times S^0, Y \times x_0) \to \pi_s^i(Y \times S^1, Y \times x_0).$$

The retraction $Y \times S^1 \to Y \times x_0$ defines the natural on Y splitting

$$\pi_s^i(Y \times S^1) = \pi_s^i(Y) \oplus \pi_s^{i-1}(Y \times S^1, Y \times x_0) = \pi_s^i(Y) \oplus \pi_s^{i-1}(Y).$$

The commutative diagram

$$\begin{tikzcd}
\pi_s^i(T \times S^1) \arrow[r, "\cong"] \arrow[d, "(f\times 1)^*"'] & \pi_s^i(T) \oplus \pi_s^{i-1}(T) \arrow[d, "f^* \oplus f^*"'] \\
\pi_s^i(W \times S^1) \arrow[r, "\cong"] & \pi_s^i(W) \oplus \pi_s^{i-1}(W)
\end{tikzcd}$$

with $i = k + 1$ implies that the homomorphism

$$(f \times 1)^* : \pi_s^{k+1}(T \times S^1) \to \pi_s^k(W \times S^1)$$

is not 0.

The inequality $\mathrm{cd}(f \times 1) < k + 1$ follows from Lemma 1.10.1. □

1.11.2 Corollary. *For any* $k > 2$ *there is a map between aspherical manifolds* $f_k : W^{k+1} \to T^k$ *with* $\mathrm{cd}(f) < k$ *and* $\mathrm{gd}(f) = k$.

Proof. We note that the map $f : W^4 \to T^3$ from (Dranishnikov & Kuanyshov 2022) induces a nontrivial isomorphism $f^* : \pi_s^3(T) \to \pi_s^3(W^4)$. We cross it with S^1 and apply Proposition 1.11.1 . Then we iterate this process until the range is a k-torus T^k. □

CHAPTER 2 FIXED-POINT-FREE INVOLUTIONS ON BOUNDARIES OF RIGHT-ANGLED COXETER GROUPS

2.1 Introductions, Right-angled Coxeter Groups

A *Coxeter System* is a pair (Γ, V) where Γ is a group, and V is a *finite* set of generators of Γ, such that the group Γ can be presented as

$$\Gamma =< v \in V : (ab)^{m(a,b)} = 1 \text{ for all } a, b \in V >$$

Where $m(a, b) \in \mathbb{Z} \cup \{\infty\}$ with the property:

1. $m(a, b) \geq 2$ for all $a, b \in V$
2. $m(a, b) = m(b, a)$ for all $a, b \in V$
3. $m(a, a) = 2$ for all $a \in V$

(Interpret the relation $(ab)^\infty = 1$ as "a, b has no relation").

The group Γ is called a *Coxeter* group. In this paper we will be primarily interested in *Right-angled* coxeter groups (RACGs), for which $m(a, b) = 2$ or ∞ for all $a, b \in V$.

Davis ((Davis 2008), (Davis 2001), (Davis 1983)) showed that for any coxeter system (Γ, V) there exists a $CAT(0)$ complex $\Sigma(\Gamma, V)$ on which Γ acts via reflections. If the context is obvious, we will denote the complex by Σ, ignoring the underlying coxeter system.

We will briefly describe Davis' construction of the complex Σ for the coxeter system (Γ, V). Consider the set $\mathcal{F}$ of subsets of V which generate a finite subgroup of Γ. It is a poset under inclusion. For every maximal element $F \in \mathcal{F}$ define P_F to be a point. Assuming that $P_{F'}$ has been defined for every $F' \supset F$, define P_F to be the cone on $\bigcup_{F' \supset F} P_{F'}$. In the last step, define $K = P_\phi = \text{Cone}\left(\bigcup_{F \neq \phi} P_F\right)$. Finally, define $\Sigma = \Gamma \times K/\sim$, where $\sim$ is defined as $(\gamma_1, x_1) \sim (\gamma_2, x_2)$ if and only if $x_1 = x_2$ and $\gamma_1^{-1}\gamma_2 \in \langle v \in V \mid x_1 = x_2 \in P_v \rangle$.

2.1.1 Theorem ((Davis 2008), Theorem 12.2.1 (credited to Gromov))**.** Suppose (Γ, V) is a right-angled coxeter system. Then Σ can be given a piecewise Euclidean CAT(0) structure.

For a given coxeter system (Γ, V) we define it's *visual boundary* $\partial\Gamma$ to be the visual boundary of its associated CAT(0) space $\Sigma(\Gamma, V)$. In other words, it is the set of geodesic rays in Σ emanating from the unit element $1 \in \Gamma$ with the topology of the uniform convergence on compact sets. We refer the reader to (Dranishnikov 1997), (Fischer 2003) or (Hosaka & Yokoi n.d.) for more details on boundaries of coxeter groups. Since our group acts on Σ via reflections, it takes geodesics to geodesics, so the action on Σ gives us an action on the boundary $\partial\Gamma$. In other words, the action of Γ on Σ gives Γ an EZ-structure.

Because of the discussion above, we can restrict ourselves to the action of Γ on Σ instead of $\partial\Gamma$.

2.2 Fixed-Point-Free Involution on the Boundary of RACGs

Suppose a group G acts on a space X. We denote the space of fixed points of X under the G-action by $\mathrm{Fix}(G, X)$.

We prove the following proposition:

1 Proposition. Given a right-angled coxeter group Γ, there exists an element $\gamma \in \Gamma$ of order two such that only finitely many points in Σ are fixed by γ.

Proof of Proposition. For a subset $S \subset \Gamma$, we say that S is *spherical* if all elements of S commute with each other. Pick a spherical subset $S \subset V$ of maximal cardinality. Let $n = |S|$. Let $S = \{a_1, a_2, \cdots, a_n\}$, and let Γ_S be the subgroup of Γ generated by S. The elements $a_1, \cdots, a_n$ act on Σ by reflections, fixing the subspaces $\Sigma_{a_1}, \cdots, \Sigma_{a_n}$ respectively. These hyperplanes intersect with each other, since all these elements commute.

Define our element γ as the product

$$\gamma = a_1 a_2 \cdots a_n.$$

γ is of order two because

$$\gamma^2 = (a_1 \cdots a_n)(a_1 \cdots a_n) = a_1^2 \cdots a_n^2 = 1.$$

By lemma 7.5.1 in (Davis 2008), since S is of maximal cardinality, $\mathrm{Fix}(\Gamma_S, \Sigma)$ is a point, let it be x_0 in Σ. Again by lemma 7.5.4 in the same book (Davis 2008), a neighborhood of x_0 is Γ_S-equivariantly a neighborhood of the origin in the canonical representation of Γ_S on $\mathbb{R}^S$. But the action of $\gamma = a_1 a_2 \cdots a_n$ corresponds to the antipodal map $z \to -z$ in $\mathbb{R}^S$, so it fixes only the origin x_0. Since Σ is CAT(0) and Γ acts via reflections, the local fixed point x_0 is the only global fixed point.

$\square$

Since the action of γ on Σ gives rise to an action of γ on $\partial\Sigma$, we obtain as a corollary, the main result of the paper:

2.2.1 Theorem. For any right-angled coxeter group Γ, there exists a fixed-point-free involution φ on $\partial\Gamma$.

Proof. The action of Γ on Σ is via reflections, so it takes geodesics to geodesics. Therefore the Γ-action on Σ gives an action of Γ on $\partial\Sigma$. By the proposition, there is an element $\gamma \in \Gamma$ such that γ has only finitely many fixed points in Σ. Since there are only finitely many fixed points, therefore no geodesics are fixed by γ, and hence the induced map φ on $\partial\Sigma$ has no fixed points. Also since γ has order two, φ is an involution. $\square$

2.3 Acknowledgement

The author would like to thank his advisor Alexander Dranishnikov for many helpful conversations regarding Coxeter groups and help with theorem 2.

CHAPTER 3 COARSE LUSTERNIK-SCHNIRELMANN CATEGORY

3.1 Introduction

In coarse geometry, dimension appeared due to Gromov in several forms (Gromov 1993, Gromov 1996). The most studied among them is the asymptotic dimension (Bell & Dranishnikov 2008) which is an important invariant in geometric group theory. Gouliang Yu proved (Yu 1998) that the finiteness of the asymptotic dimension of a group Γ implies the Novikov Higher Signature conjecture for Γ and most of the satellite conjectures.

Proving that the asymptotic dimension of a certain group or a class of groups is finite is often a great challenge. It was proven for hyperbolic groups (Gromov 1987, Roe 2005), nilpotent groups (Bell & Dranishnikov 2008), solvable groups (Dranishnikov & Smith 2006), arithmetic groups, (Ji 2004) and for mapping class groups (Bestvina, Bromberg & Fujiwara 2010). The next challenge are the groups $\mathrm{Out}(F_n)$ and Helly groups (Bandelt & Prisner 1991, Chalopin, Chepoi, Genevois, Hirai & Osajda 2025).

We note that Gromov's definition of asymptotic dimension was a translation of Lebesgue's definition of the covering dimension to the language of coarse geometry. In this paper we do a similar thing with another numerical invariant from classical topology, the Lusternik-Schnirelmann category $\mathrm{cat}(X)$ (LS-category for short). In topology, the LS-category is a lower bound for dimension, $\mathrm{cat}(X) \leq \dim(X)$, some generalization work has been done in (Margolis 2024, Srinivasan 2015). Our main result is a similar inequality

$$\text{c-cat}(X) \leq \mathrm{asdim}(X)$$

for certain classes of groups and metric spaces (Theorem 3.5.7, 3.5.6). Thus in the open problems about the finiteness of asymptotic dimension perhaps the first step would be to try to prove the finiteness of the coarse LS-category.

If a classifying space $B\Gamma$ of a group Γ is compact, then its universal cover $E\Gamma$ with lifted geodesic metric from $B\Gamma$ is coarsely equivalent to Γ with a word metric. In that case $E\Gamma$ is a proper metric space. The LS-category in the category of locally compact metric spaces and proper maps was defined and studied (Ayala, Domínguez Murillo, Márquez Pérez & Quintero 1992) before. We prove the following comparison result in theorem 3.3.2:

3.1.1 Theorem. For a geometrically finite group Γ we have the following inequality

$$\text{p-cat}(E\Gamma) \leq \text{c-cat}(E\Gamma).$$

3.2 The Coarse Homotopy Category

Let us give a brief description of coarse maps and coarse homotopies, when restricted to metric spaces. The following can be generalized for general coarse structures, the interested reader is encouraged to look at (Roe 2003, Mitchener, Norouzizadeh & Schick 2020) for a more detailed overview.

3.2.1 Definition. Let X and Y be metric spaces. A (not necessarily continuous) function $f : X \to Y$ is called *controlled*, or *bornologous* if for every $r > 0$ there exists a $S > 0$ such that

$$d(x, x') < r \implies d(f(x), f(x')) < S$$

for all $x, x' \in X$. The function f is called *proper* if for any bounded subset $B \subset Y$, the preimage $f^{-1}(B) \in X$ is bounded. f is called *coarse* if it is both controlled and proper.

3.2.2 Definition. Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be any function. A function $f : X \to Y$ between metric spaces is called *ρ-bornologous* if it satisfies

$$d(f(x), f(x')) \leq \rho(d(x, x'))$$

for all $x, x' \in X$. Clearly, f is controlled (or bornologous) if f is ρ-bornologous for some ρ.

3.2.3 Definition. Let $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be two functions going to infinity, and let $f : X \to Y$ be a function between metric spaces. f is called (ρ_1, ρ_2)-*coarse* if it satisfies

$$\rho_1(d(x,x')) \leq d(f(x), f(x')) \leq \rho_2(d(x,x'))$$

for all $x, x' \in X$. It is straightforward to check that a function between metric spaces is coarse if and only if it is (ρ_1, ρ_2)-coarse for some functions ρ_1, ρ_2 going to infinity.

3.2.4 Definition. Two functions $f, g : X \to Y$ between metric spaces are *close* or *uniformly bounded distance apart* if there exist a constant $M \in \mathbb{R}_+$ such that $d(f(x), g(x)) \leq M$ for all $x \in X$. f is called *bounded* if it is close to a constant map.

Two metric spaces X, Y are called *coarsely equivalent* if there exist maps $f : X \to Y$ and $g : Y \to X$ such that the compositions $f \circ g$ and $g \circ f$ are close to their respective identity maps.

3.2.1 Example. Standard Examples of coarse maps would be

- The floor function $\lfloor . \rfloor : \mathbb{R} \to \mathbb{Z}$, and the inclusion $\mathbb{Z} \to \mathbb{R}$. This shows us $\mathbb{R}$ and $\mathbb{Z}$ are coarsely equivalent.
- In general a similar argument shows us that the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ is coarsely equivalent to $\mathbb{R}^n$.
- Let M be a complete simply-connected Riemannian manifold of non-positive sectional curvature. For a point $p \in M$, The exponential map $\exp : T_pM \to M$ is a distance-increasing diffeomorphism. The inverse $\log : M \to T_pM$ is therefore a coarse map.

3.2.5 Definition. Any coarse map $f : \mathbb{R}_+ \to X$ will be called a *coarse ray*, and a coarse map $f : \mathbb{R}_+ \to X$ is called a *coarse* $\mathbb{R}_+$*-basepoint* if the map f is coarsely equivalent to a quasi-geodesic ray.

Note that our notion of a coarse $\mathbb{R}_+$-basepoint a slightly more restrictive than the notion of a $\mathbb{R}_+$-basepoint in (Mitchener et al. 2020). The authors there consider any coarse ray to be a $\mathbb{R}_+$-basepoint. Every proof in this paper goes through with that definition, barring 3.2.4, 3.5.8.

3.2.1 Coarse Homotopy

The purpose of this sub-section is to define the notion of homotopy in the coarse category, as defined in (Mitchener et al. 2020). These homotopies have to end eventually, but the end time will be allowed to depend on the given point in the metric space (and to go to infinity as one goes to infinity). These will be measured by coarse maps $p : X \to \mathbb{R}_+$, which are sometimes called "base-point projections". Note that if we fix a base-point $x_0 \in X$, there is a natural choice for the projection $p_0 : X \to \mathbb{R}_+$ given by $p_0(x) = d(x, x_0)$. p_0 is also called the *standard base-point projection*.

Before defining coarse homotopies, we need to define cylinders.

3.2.6 Definition. Let X be a metric space, and let $p : X \to \mathbb{R}_+$ be a coarse map. We define the p-cylinder

$$I_p X = \{(x, t) \in X \times \mathbb{R}_+ \mid t \leq p(x)\}.$$

If we have a pointed metric space (X, x_0), we have a natural choice of coarse map, $p(x) = d(x, x_0)$. In this case we will omit the p and denote $I(X) = I_p(X)$.

We have inclusions $i_0, i_1 : X \to I_p X$ defined by the formulas $i_0(x) = (x, 0)$ and $i_1(x) = (x, p(x))$. The canonical projection $q : I_p X \to X$ is a coarse map, and identities $q \circ i_0 = q \circ i_1 = 1_X$ clearly hold.

3.2.7 Definition. Let X, Y be metric spaces. A *coarse homotopy* is a coarse map $H : I_p X \to Y$ for some coarse map $p : X \to \mathbb{R}_+$.

We call coarse maps $f, g : X \to Y$ *coarsely homotopic* if there is a coarse map $p : X \to \mathbb{R}_+$ and a coarse homotopy $H : I_p X \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$.

Let $f : X \to Y$ be a coarse map between metric spaces. We call the map f a *coarse homotopy equivalence* if there is a coarse map $g : Y \to X$ such that the compositions $g \circ f$ and $f \circ g$ are coarsely homotopic to the identities 1_X and 1_Y respectively.

We mention a few properties of coarse homotopies, proofs can be found in (Mitchener et al. 2020).

- If two coarse maps $f, g : X \to Y$ are close, then they are coarsely homotopic.
- Coarse homotopy is an equivalence relation.
- For path-metric spaces, the choice of the map $p : X \to \mathbb{R}_+$ does not matter:

  3.2.2 Lemma. (Mitchener et al. 2020, lemma 2.6) Let X be a path-metric space. For $x_0 \in X$, let $p_0 : X \to \mathbb{R}_+$: $x \mapsto d(x, x_0)$ be the standard base-point projection. Let $q : X \to \mathbb{R}_+$ be any coarse map. Then any coarse homotopy $H : I_q X \to Y$ between $f, g : X \to Y$ gives rise to a coarse homotopy $\bar{H} : I_{p_0} X \to Y$ between f and g.

3.2.3 Example. Let M be a complete simply-connected Riemannian manifold of non-positive sectional curvature and let $p \in M$. The exponential map $exp : T_p M \to M$ is distance-increasing diffeomorphism, hence the inverse $log : M \to \mathbb{R}^n$ is a coarse map. Although exp is not coarse (so log is not necessarily a coarse equivalence), one can construct a *coarse homotopy inverse* of log by using a radial shrinking function on $\mathbb{R}^n$ (Mitchener et al. 2020, Example 2.7). This proves that log is a coarse homotopy equivalence. In particular, $\mathbb{R}^n$ and the hyperbolic space $\mathbb{H}^n$ are coarsely homotopy equivalent.

3.2.8 Definition. A metric space X is *coarsely path-connected* if any two coarse $\mathbb{R}_+$-basepoints are coarsely homotopic to each other.

Note that the notion of *coarsely path-connectedness* is very similar to the notion of semistability at ∞ in the group-theory world.

3.2.2 Semistability at Infinity

Firstly recall the notion of a Freudenthal end of a space X. The *space of ends of* X, denoted by $\mathscr{F}(X)$ is the inverse limit $\mathscr{F}(X) = \varprojlim \pi_0(X - K)$ where K ranges over the family of compact subsets of X, and π_0 stands for the set of connected components. Roughly speaking, these are the "Connected components at infinity". If our space is a group Γ with the word metric, it is known that $\mathscr{F}(\Gamma)$ can have either $0, 1, 2$ or ∞ elements. A group is called *semistable at* ∞ if any two geodesic rays in the same end of the group can be properly homotoped to one another. It is known that any 0 or 2-ended group is semistable at ∞, but the other two cases are open problems.

3.2.4 Claim. For a discrete group Γ with the word metric, Γ is 1-ended semistable at ∞ if and only if Γ is coarsely path-connected.

Proof. Consider X to be the Cayley 2-complex of the group Γ with some generating set, which is coarsely equivalent to Γ. Suppose Γ is 1-ended and semistable at ∞. Let $\alpha, \beta : \mathbb{R}_+ \to X$ be two coarse $\mathbb{R}_+$-basepoints.

Since X is a path-metric space, We can get $\alpha', \beta' : \mathbb{R}_+ \to X$ geodesic rays that are coarsely equivalent to α, β.

Since Γ is semistable at ∞, α', β' are properly homotopic to each other. Since $\mathbb{R}_+$ is combable, using proposition 4 we conclude that α', β' are coarsely homotopic. Since α', β' are coarsely equivalent to α, β, we are done!

For the converse, suppose Γ is coarsely path-connected. Clearly Γ has to be 1-ended. Suppose $p, q : \mathbb{R}_+ \to \Gamma$ are two geodesics. Since Γ is coarsely path-connected, there exists a coarse homotopy $H_c : I(\mathbb{R}_+) \to \Gamma$. We can give $I(\mathbb{R}_+)$ a uniformly contractible simplicial complex structure, so we can use lemma 3.3.1, to get a continuous coarse map $H' : I_p(\mathbb{R}_+) \to \Gamma$. Using H', we can get a proper homotopy between p, q. □

3.2.3 Coarse Lusternik-Schnirelmann Category

One of the classical homotopy invariants in the category **Top** of topological spaces is the Lusternik-Schnirelmann category $\mathrm{cat}(X)$, or LS-category in short. The (reduced) $\mathrm{cat}(X)$ is defined as the smallest number k such that there is an open covering $\{U_i\}_{0 \le i \le k}$ of X with the property that each inclusion map $U_i \hookrightarrow X$ is nullhomotopic. We aim to define an analog of this for the coarse homotopy category.

In the coarse category, instead of being nullhomotopic, we need to define a notion of sets being "small":

3.2.9 Definition. Let X be a metric space. A subset $A \subseteq X$ is called *coarsely categorical* if there exist coarse maps $\alpha : \mathbb{R}_+ \to X$ and $j : A \to \mathbb{R}_+$ such that the following diagram commutes up to coarse homotopy:

$$\begin{tikzcd} A \arrow[rr, hook] \arrow[dr, "j"'] & & X \\ & \mathbb{R}_+ \arrow[ur, "\alpha"'] & \end{tikzcd}$$

Where the top horizontal map is the canonical inclusion of A into X.

The map $\alpha : \mathbb{R}_+ \to X$ is called an $\mathbb{R}_+$-base-point. Now we can define coarse LS-category:

3.2.10 Definition (coarse LS-category)**.** Let X be a metric space. The (reduced) coarse LS-category of X, denoted by c-cat(X) is the least number k such that there exists a covering $\{U_i\}_{0\leq i\leq k}$ of X by $k+1$ coarsely categorical sets.

It is clear from the definition that for a metric space X, c-cat$(X) = 0$ if and only if X is coarsely homotopy equivalent to $\mathbb{R}_+$. The following two results show that c-cat is a coarse homotopy invariant.

3.2.5 Lemma. If there exist coarse maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g : Y \to Y$ is coarsely homotopic to the identity 1_Y, then c-cat$(Y) \leq$ c-cat(X).

Proof. Suppose c-cat$(X) \leq k$, let $\{U_i\}_{i=0}^k$ be a covering of X by coarsely categorical sets. For each i, define $V_i = g^{-1}(U_i)$ to be the preimage of U_i under g. We claim that each V_i is coarsely categorical in X. Since $f \circ g$ is coarsely homotopic to the identity, choose a homotopy $H : I_p(Y) \to Y$ such that $H \circ i_0 = 1_Y$ and $H \circ i_1 = f \circ g$. Via this homotopy, each V_i is homotoped into

$$(f \circ g)(V_i) = (f \circ g)(g^{-1}(U_i)) = f(U_i).$$

But since each U_i is coarsely categorical in X, their image $f(U_i)$ in Y is coarsely categorical as well. Composing these two homotopies, we get the result. □

2 Proposition. If X, Y are metric spaces which are coarsely homotopy equivalent, then c-cat$(X) =$ c-cat(Y).

Proof. This follows from the above lemma 3.2.5. If X, Y are coarsely equivalent, then there are coarse maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are coarsely homotopic to identity maps. Using lemma 3.2.5 we can say c-cat$(X) \leq$ c-cat(Y) and c-cat$(Y) \leq$ c-cat(X), therefore both quantities are equal. □

Since c-cat is a coarse homotopy-invariant, we can define $\text{c-cat}(\Gamma) = \text{c-cat}(X)$ for any proper geodesic metric space X on which Γ acts properly and cocompactly via isometries.

3.2.6 Example. 1. $\text{c-cat}(X) = 0$ if and only if X is coarsely homotopy equivalent to $\mathbb{R}_+$.

2. $\text{c-cat}(\mathbb{R}^n) = 1$ for all $n > 0$. We can cover $\mathbb{R}^n$ with two halves $A = [0, \infty) \times \mathbb{R}^{n-1}$ and $B = [0, \infty) \times \mathbb{R}^{n-1}$. A can be coarsely deformed to the positive ray, and B the negative ray. $\text{c-cat}(\mathbb{R}^n) > 0$ because of lemma 3.

3. Because of the above, $\text{c-cat}(M) = 1$ for any complete simply-connected Riemannian manifold of non-positive sectional curvature, and $\text{c-cat}(\mathbb{Z}^n) = 1$.

4. For the infinite binary tree T_2, no two distinct geodesic rays are coarsely homotopic. Hence $\text{c-cat}(T_2) = \infty$.

3.3 Coarse LS-category vs proper LS-category

In (Ayala et al. 1992) the authors explored the idea of proper LS category p-cat, which is quite similar to our notion of c-cat, where instead of requiring maps to be coarse, they require all maps to be proper and continuous. This is slightly less restrictive, so as we'll see from the discussion below, for a lot of reasonable spaces X, we have the inequality $\text{p-cat}(X) \leq \text{c-cat}(X)$.

Before we go into the definitions, let us first prove a lemma that allows us to "upgrade" coarse (not necessarily continuous) maps to proper continuous maps.

3.3.1 Lemma. Suppose we have a coarse map $f : X \to Y$ between metric spaces X, Y with the properties that X is a finite dimensional simplicial complex whose simplices are uniformly bounded, and Y is uniformly contractible. Then there exists a coarse continuous map $g : X \to Y$ which is in bounded distance to f.

Proof. First consider $X^{(0)}$, the zero-skeleton of X. We can define g^0 on $X^{(0)}$ to be the restriction of f, and it is continuous and coarse (since $X^{(0)}$ is discrete). Inductively, suppose we have constructed a continuous function $g^k : X^{(k)} \to Y$ on the k-skeleton of X which is coarse and close to f.

Using the uniform contractibility of Y, we can extend the map g to a larger map g^{k+1} as shown by the dashed arrow:

$$\begin{tikzcd} X^{(k)} \arrow[r, "g^k"] \arrow[d] & Y \\ X^{(k+1)} \arrow[ur, dashed, "g^{k+1}"'] & \end{tikzcd} \tag{3-1}$$

Locally this is done by "filling in" the simplexes using contractions, and because of uniform contractibility, we know that these simplexes can be filled in while still being uniformly close to f. Because X is a finite dimensional simplicial complex, this process terminates eventually, and we get our continuous map $g = g^n$ where $n = \dim X$. □

Now we can introduce the notion of proper LS-category as done in (Ayala et al. 1992). We will be working in the category $\mathfrak{B}_\infty$ of non-compact T_2-locally compact spaces, and proper continuous maps as morphisms. Let $X \in \mathfrak{B}_\infty$ be such a space.

Define a closed subset $C \subseteq X$ to be *properly deformable to* $\mathbb{R}_+$ if there exists a diagram in $\mathfrak{B}_\infty$:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "r"] & X \\ \mathbb{R}_+ \arrow[ur, bend right, "\alpha"'] & \end{tikzcd}$$

which commutes up to proper homotopy.

3.3.1 Definition. Given a space X in $\mathfrak{B}_\infty$, $A \subseteq X$ is said to be properly categorical in X if there is a closed neighborhood of A properly deformable to $\mathbb{R}_+$ in X.

An open covering $\{U_\alpha\}$ of X is said to be properly categorical if each U_α is properly categorical in X.

The proper Lusternik-Schnirelmann category (or proper LS category) of X, denoted by p-cat(X) is the least number n such that X admits a properly categorical open covering with n elements. If no finite properly categorical covering exists then p-cat$(X) = \infty$.

Using our lemma 3.3.1 we can compare our notions of c-cat(X) and p-cat(X) when our space X is nice:

3 Proposition. Let X be a uniformly contractible finite dimensional simplicial complex, with a metric such that the simplices are of bounded diameter. Then

$$\text{p-cat}(X) \leq \text{c-cat}(X).$$

Proof. Suppose $\text{c-cat}(X) = n$, and let $\{U_1, U_2, \cdots, U_{n+1}\}$ be a coarsely categorical cover of X. Define V_i to be the simplicial neighborhood of U_i in X. Since simplices in X are of bounded diameter, and X is uniformly contractible, we can extend the coarse homotopies of U_i to V_i's, and therefore $\{V_1, \cdots, V_{n+1}\}$ is a coarsely categorical cover of X as well.

We can give a simplicial complex structure to each $I(V_i)$ such that the simplices are uniformly bounded. Consider the coarse homotopy $H_i : I(V_i) \to X$. Using lemma 3.3.1 we can say that each V_i is properly categorical. Hence $\{\text{int}(V_1), \cdots, \text{int}(V_{n+1})\}$ gives us a properly categorical open cover of X, so $\text{p-cat}(X) \leq n$. □

A good class of examples of such simplicial complexes are universal covers of groups which have a finite $K(G,1)$-complex. These are the so-called *geometrically finite* groups. Hence we have:

3.3.2 Theorem. For a geometrically finite group Γ, we have the inequality

$$\text{p-cat}(\Gamma) = \text{p-cat}(E\Gamma) \leq \text{c-cat}(E\Gamma) = \text{c-cat}(\Gamma).$$

3.3.3 Example.

1. $\text{p-cat}(\mathbb{R}^n) = 1$ for all $n > 0$.

2. Let X be the euclidean plane without a strip around the negative x-axis: $X = \mathbb{R}^2 \setminus (-\infty, 0) \times (-1, 1)$. Then X is properly deformable to the positive x-axis, so $\text{p-cat}(X) = 0$. But X is coarsely equivalent to the plane, hence $\text{c-cat}(X) = 1 > \text{p-cat}(X)$.

3. Let T be an embedding of the binary tree T_2 inside $[0,1] \times \mathbb{R}_+$. $\text{p-cat}(T) = \infty$, but since T is coarsely equivalent to $\mathbb{R}_+$, $\text{c-cat}(T) = 0 < \text{p-cat}(T)$.

3.4 Combable Spaces

In this section we will define the notion of combable spaces. Most of our results in the subsequent sections will be about these spaces. Interested readers can check (Kato 2000) for a more detailed description of combability and bicombability.

Let X be a metric space. For convenience, from now on $\mathbb{N}$ will represent the set of nonnegative integers.

3.4.1 Definition. ((Engel & Wulff 2021), definition 2.4) By a *combing* on X starting at a point $p \in X$ we mean a map

$$C : X \times \mathbb{N} \to X$$

such that

1. $C(x, 0) = p = C(p, n)$ for all $x \in X$ and $n \in \mathbb{N}$.
2. for each bounded subset $K \subset X$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in K$ we have $C(x, n) = x$.
3. C is a controlled map.

Moreover, C is called *proper* if for any bounded subset $K \subset X$ there is a bounded subset $L \subset X$ and an $N \in \mathbb{N}$ such that $C^{-1}(K) \subset L$ for all $n \geq N$.

If a given space can be equipped with a combing then it is called *combable*.

In a similar vein, one can define a bicombing:

3.4.2 Definition. Let X be a metric space. A *bicombing* on X is a map

$$C : X \times X \times \mathbb{N} \to X; \quad C_p(x, n) := C(p, x, n) \quad \forall p, x \in X, n \in \mathbb{N}$$

which satisfies

1. $C_p(x, 0) = C_p(p, n) = p$ for all $p, x \in X$, $n \in \mathbb{N}$.

2. For each $p \in X$ and $K \subset X$ bounded, there exist $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in X$, $C_p(x,n) = x$.

3. C is a controlled map.

One can regard $C(x,-)$ or $C_p(x,-)$ as a path from the point p at time 0 to the point x which eventually becomes constant (encapsulated in point 2). The map C being *controlled* implies that these paths are uniformly coarse, and they have the so-called "fellow-travelling property". If G is a finitely generated group endowed with the word metric, then this notion of a (bi)combing is equivalent to the usual notion used in geometric group theory literature (they are sometimes called synchronous (bi)combings or also bounded (bi)combings).

If we consider the singleton bounded set $K = \{x\}$ for property 2, there exist a smallest $N =: N_p(x)$ such that $C_p(x,n) = x$ for all $n \geq N$. So each path $C_p(x,-)$ is of "length" $N_p(x)$.

We prove a "coarse" version of uniform contractibility for geodesic metric spaces which are combable.

3.4.3 Definition $(*)$**.** A metric space X has property $(*)$ if there exist functions $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$ going to infinity, such that for any r-ball $B_p(r)$ around any point $p \in X$, we have a (not necessarily continuous) map

$$h : I(B_p(r)) \to X$$

such that

1. $h \circ i_0(x) = x$ and $h \circ i_1(x) = p$ for all $x \in B_p(r)$.

2. h is (ρ_1, ρ_2)-coarse. (see Definition 3.2.3)

One can interpret definition 3.4.3 as a coarse version of uniform contractibility, where the balls are uniformly coarsely contractible.

3.4.1 Lemma. Let X be a bicombable path-metric space. Then X has $(*)$.

Proof. Suppose our space X has a bicombing

$$C : X \times X \times \mathbb{N} \to X.$$

Because of the reasons discussed above, the map $\gamma : X \to \mathbb{R}_+$ defined by $\gamma(x) = N_p(x)$ is a coarse map. We can now define a coarse homotopy

$$h' : I_\gamma(B_p(r)) \to X$$

$$h'(x,t) = C_p(x, \lfloor N_p(x) - t \rfloor)$$

where $\lfloor . \rfloor$ denotes the floor function. Our function h' has the property $h' \circ i_0(x) = C_p(x, N_p(x)) = x$, and $h' \circ i_1(x) = C_p(x, 0) = p$. Thus property 1 is satisfied. Property 2 follows from the fact that C is controlled.

Since X is a path-metric space, by lemma 3.2.2 we get the result. □

For the rest of this section, we will consider cases where proper maps can be "upgraded" to coarse maps, so we get another upper bound for c-cat. First, we need to define the so-called *shrinking map*.

3.4.4 Definition. Suppose we have a pointed metric space (X, p) which admits a combing $C : X \times \mathbb{N} \to X$ for the base-point p. Suppose $\rho : \mathbb{N} \to \mathbb{N}$ is any map such that $\rho(t) \leq t$ for all $t \in \mathbb{N}$. Then one can define the *shrinking map* $\mathrm{Sh}_\rho : X \to X$ as

$$\mathrm{Sh}_\rho(x) = C\left(x, \rho(N_x)\right) \tag{3-2}$$

Here (and from here onwards) will use the notation $\|x\| = d(x, p)$ for any point $x \in X$, whenever the base point p is obvious from context.

Note that in such a case, for each $x \in X$ one can go from x to $\mathrm{Sh}_\rho(x)$ "along the combing C", so we get the following lemma:

3.4.2 Lemma. For a combable pointed metric space (X, p) and a coarse map $\rho : \mathbb{N} \to \mathbb{N}$ such that $\rho(t) \leq t$ for all $t \in \mathbb{N}$, the shrinking map Sh_ρ defined in equation 3-2 is coarsely homotopic to the identity map.

Using this lemma, we can prove the following generalization of an earlier result by Thomas Weighill (Weighill 2019):

4 Proposition. Let (X, p) be a combable pointed metric space, and Y be any proper metric space. If two continuous coarse maps $f, g : X \to Y$ are properly homotopic, then f, g are coarsely homotopic.

Proof. Let $h : X \times [0, 1] \to Y$ be a proper homotopy from f to g. We can extend h by defining a new function $h' : I_p X \to Y$:

$$h'(x,t) = \begin{cases} h\left(x, \frac{t}{\|x\|}\right), & x \neq p \\ p & x = p \end{cases} \tag{3-3}$$

We will now construct a decreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$. Since Y is a proper metric space and $h' : I_p X \to Y$ is a continuous map, we can find integers L_k such that

$$d((x,t),(x',t')) \leq \frac{1}{L_k} \implies d(h'(x,t), h'(x',t')) \leq 1$$

for $x \in B_p(k)$ and $t \in [0, k]$. Also we can choose L_k's to be increasing, and all greater than one.

Define $\rho : \mathbb{N} \to \mathbb{N}$ as the unique map that maps $\{0, \cdots, L_1\}$ "linearly" to $\{0, 1\}$, $\{L_1, \cdots, L_1 + 2L_2\}$ to $\{1, 2\}$, $\{L_1 + 2L_2, \cdots, L_1 + 2L_2 + 3L_3\}$ to $\{2, 3\}$ and so on. Figure 3-1 describes the map.

Having defined ρ, let us define $H : I_p X \to Y$ as

$$H(x,t) = h'\left(\mathrm{Sh}_\rho(x), t\frac{\rho(\|x\|)}{\|x\|}\right)$$

Now we can check that

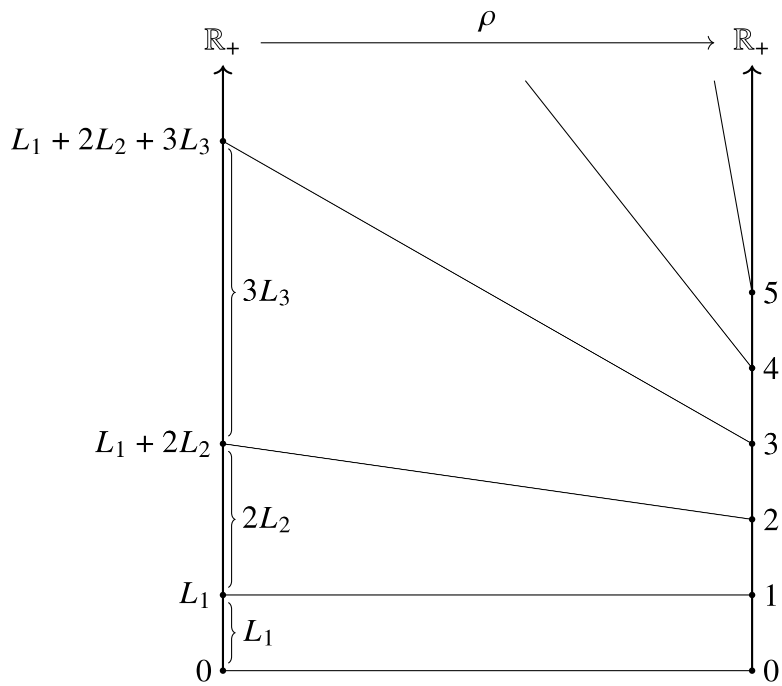

Figure 3-1. A visual description of the map $\rho : \mathbb{N} \to \mathbb{N}$

3.4.3 Claim.

$$d((x,t),(x',t')) \le 1 \implies d(H(x,t),H(x',t')) \le 1$$

for all $(x,t),(x',t') \in I_p X$.

Assuming the claim, it is easy to see that H is a coarse homotopy between $H \circ i_0$ and $H \circ i_1$. But

$$H \circ i_0(x) = h'(\mathrm{Sh}_\rho(x), 0) = f(\mathrm{Sh}_\rho(x)) = (f \circ \mathrm{Sh}_\rho)(x)$$

and

$$\begin{aligned} H \circ i_1(x) &= h'(\mathrm{Sh}_\rho(x), \rho(\|x\|)) \\ &= h'(\mathrm{Sh}_\rho(x), \left\|\mathrm{Sh}_\rho(x)\right\|) \\ &= g(\mathrm{Sh}_\rho(x)) = (g \circ \mathrm{Sh}_\rho)(x). \end{aligned}$$

So our constructed H is a coarse homotopy between $f \circ \mathrm{Sh}_\rho$ and $g \circ \mathrm{Sh}_\rho$. Using lemma 3.4.2, we get our required result. □

Now we can prove our earlier claim.

proof of claim 3.4.3. Let's assume we are using the supremum metric on $I_p X$. We can prove the same for other metrics similarly.

Suppose $d((x,t),(x',t')) \leq 1$. Therefore $d(x,x') \leq 1$ and $d(t,t') \leq 1$. We can assume both points (x,t) and (x',t') lie in

$$\{(x,t) \in I_p X \mid L_1 + \cdots + (k-1)L_{k-1} \leq \|x\| \leq L_1 + \cdots + kL_k\}$$

for some $k \in \mathbb{N}$. For convenience, define $y = \mathrm{Sh}_\rho(x)$, $y' = \mathrm{Sh}_\rho(x')$, $s = t\frac{\rho(\|x\|)}{\|x\|}$, $s' = t'\frac{\rho(\|x\|')}{\|x\|'}$. Therefore, $d(x,x') \leq 1$ implies

$$d(y,y') = d(\mathrm{Sh}_\rho(x), \mathrm{Sh}_\rho(x')) \leq \frac{1}{kL_k} \leq \frac{1}{L_k}. \tag{3-4}$$

(Even if both points $(x,t),(x',t')$ do not lie in such a set, we can take the smallest k such that one of the points lie in such a set, and the inequality is still true.) Similarly one can argue that (assuming $\|x\| \leq \|x'\|$)

$$\begin{aligned} d(s,s') = d\left(t\frac{\rho(\|x\|)}{\|x\|}, t'\frac{\rho(\|x\|')}{\|x\|'},\right) &\leq \frac{\rho(\|x\|)}{\|x\|} d(t,t') \\ &\leq \frac{k}{L_1 + \cdots + kL_k} d(t,t') \\ &\leq \frac{1}{L_k} d(t,t') \leq \frac{1}{L_k}. \end{aligned} \tag{3-5}$$

Thus if x, x' satisfy $L_1 + \cdots + (k-1)L_{k-1} \leq \|x\| \leq L_1 + \cdots + kL_k$, then combining inequalities 3-4 and 3-5 we get

$$\begin{aligned} d((x,t),(x',t')) \leq 1 &\implies d((y,s),(y',s')) \leq \frac{1}{L_k} \\ &\implies d(h'(y,s), h'(y',s')) \leq 1 \\ &\implies d(H(x,t), H(x',t')) \leq 1 \end{aligned}$$

This is independent of k, so is true for all choices of x, x', t, t'. Hence proved. □

Now using our proposition 4 we can now establish the following upper bound for c-cat:

3.4.4 Theorem. Let (X, p) be a pointed metric space. Then c-cat$(X) \leq k$, where k is the least number such that X can be covered by k subsets each of which admits a combing, and is properly contractible to a ray.

3.5 Asymptotic dimension and Coarse Category

The main result of this section is the following:

3.5.1 Theorem. For a bicombable proper geodesic metric space X which is coarsely semistable at infinity, we have the inequality c-cat$(X) \leq$ asdim(X).

For the majority of this section and what follows it, we will be considering based metric spaces (X, p), i.e., a metric space X and a base-point $p \in X$. But before we can prove the theorem, we need to develop some machinery.

3.5.1 Dispersed sets and families

Let's define some notions of "dispersed" sets, defined in (Bell & Dranishnikov 2005).

From here on, we will use the notation

$$B_p(r) = \{x \in X \mid d(x, p) < r\}$$

and

$$A_p(r, R) = B_p(R) \setminus B_p(r)$$

for any metric space X, $p \in X$ and real numbers $R > r > 0$.

3.5.1 Definition. Given a based metric space (X, p) and a discrete subset $U \subset X$, U is called *dispersed* if the function

$$\partial(r) = \inf\{d(x_1, x_2) : x_1 \neq x_2 \in U \setminus B_p(r)\}$$

goes to infinity ($lim_{r\to\infty}\partial(r) = \infty$).

3.5.2 Definition. Given a based metric space (X, p) and a family of subsets $\mathcal{U}$ of X, we say the family is dispersed if

1. Elements of $\mathcal{U}$ are bounded and pairwise disjoint.

2. The function

$$\partial(r) = \inf\{d(D_1, D_2) \mid d(p, D_i) > r \text{ and } D_1 \neq D_2 \in \mathcal{U}\}$$

   goes to infinity ($lim_{r\to\infty}\partial(r) = \infty$).

For a combable geodesic metric space X, dispersed families can be coarsely homotoped into dispersed sets:

3.5.2 Lemma. suppose X is a bicombable geodesic metric space, and let $\mathcal{U}$ be a dispersed family of subsets of X. Then the union $U = \bigcup_{a\in\mathcal{U}} a$ is coarsely homotopic to a dispersed set in X.

Proof. For each $A \in \mathcal{U}$, choose a base-point $e(A) \in A$. Since X is bicombable and A is bounded, we can use lemma 3.4.1 to get a homotopy $H_A : A \times I \to X$ to the point $e(A)$. We can paste these individual maps to get a homotopy $H : I_d U \to X$ from $U = \bigcup_{A\in\mathcal{U}} A$ to $V = \bigcup_{A\in\mathcal{U}}\{e(A)\}$. We can paste them together nicely because of property 3 of lemma 3.4.1. □

3.5.3 Lemma. Suppose X is a bicombable proper geodesic metric space which is coarsely semistable at infinity, and $\mathcal{U} \subset X$ is a dispersed set. Then U is coarsely categorical.

Before we prove lemma 3.5.3 we need to prove a technical lemma:

3.5.4 Lemma. Let X be a proper geodesic metric space which is coarsely path-connected, and choose an $\mathbb{R}_+$-base-point $\alpha : \mathbb{R}_+ \to X$, denote $p = \alpha(0)$. Suppose a subset $A \subset X$ is dispersed. Then there are functions $\gamma, \rho : \mathbb{R}_+ \to \mathbb{R}_+$ having the following properties:

1. $\gamma(R) > R$ for all $R > 0$.

2. Fix any $R > 0$. For each point $x \in A \setminus B_p(\gamma(R))$, we can define a path $h_x : [0, k_x] \to X$ which have the properties:

(a) $h_x(0) = x, h_x(k_x) \in \mathrm{Im}(\alpha)$.

(b) $\mathrm{Im}(h_x) \cap B_p(R) = \phi$.

(c) h_x is ρ-bornologous.

(d) $k_x \leq 2\|x\|$.

What lemma 3.5.4 states is that for any radius $R > 0$, there is a larger radius $\gamma(R) = S > 0$ such that all points in A outside the S-ball around p can be joined to the $\mathbb{R}_+$-base-point α via a path that does not intersect $B_p(R)$. Furthermore, we can choose these paths to be nice: uniformly ρ-bornologous, and at most $2\|x\|$-long. An illustration is given in figure 3-2.

proof of lemma 3.5.4. First we prove such a path exists, then we prove the stronger conditions of the path. Suppose such a path cannot exist. So there exists a radius R such that there exist points arbitrarily far away from p such that all paths joining those points to the ray α passes through $B_p(R)$. Choose a sequence of such points $\{x_i \in X\}_{i\in\mathbb{N}}$ such that $r_i = \|x_i\| \nearrow \infty$.

Note that for each x_i for $i > 1$, we can choose a geodesic $g_i : [0, r_i] \to X$ such that $g_i(0) = p$ and $g_i(r_i) = x_i$. For any $j \in \{1, 2, \cdots, i-1\}$ define $x_i^j = g_i(r_j)$, so we have $\left\|x_i^j\right\| = r_j$. For convenience, define $x_i^i = x_i$. Since X is a proper metric space, the spheres $S_p(r_j)$ are compact, hence the set $\{x_i^j \mid i \geq j\} \subset S_p(r_j)$ has an accumulation point $y_j \in S_p(r_j)$. Suppose $d(y_j, y_{j+1}) = d_j$. Construct geodesics $\beta_j : [0, d_j] \to X$ from y_j to y_{j+1}. Pick a geodesic $\gamma : [0, r_1] \to X$ from p to y_1. Now we can define a map $\beta : \mathbb{R}_+ \to X$ as

$$\beta(t) = \begin{cases} \gamma(t) & \text{if } t \leq r_1 \\ \beta_j(t - (r_1 + \cdots + r_{j-1})) & \text{if } r_1 + \cdots + r_{j-1} < t \leq r_1 + \cdots + r_j \end{cases} \tag{3-6}$$

Note that β is concatenation of γ and the β_j's. Since β is piecewise geodesic, the map β is bornologous. It is easy to see that β is proper as well, so β is a coarse map.

Since X is coarsely path-connected, we can define a coarse homotopy $H : I_d\mathbb{R}_+ \to X$ such that $H \circ i_0 = \alpha$ and $H \circ i_1 = \beta$. But this is a contradiction!

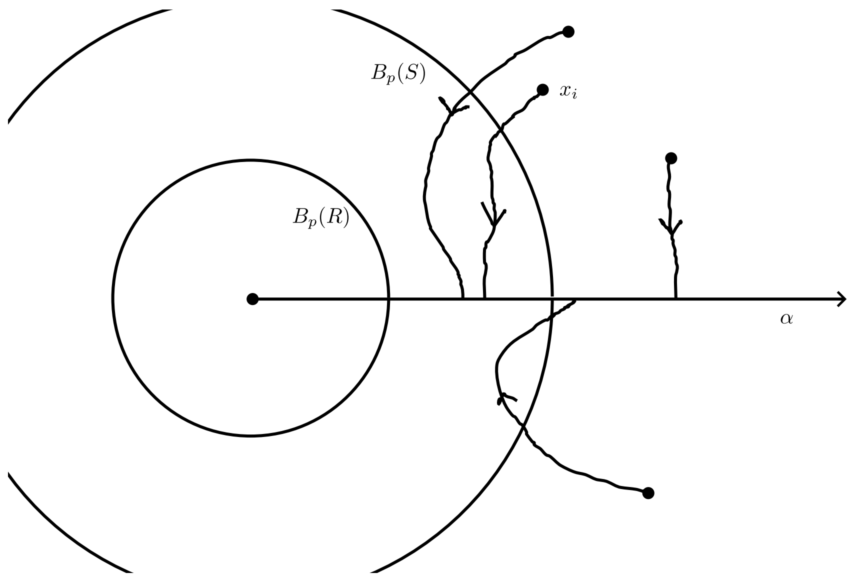

Figure 3-2. Illustration of lemma 3.5.4

Indeed, since H is a coarse map, in particular it is proper. So there exists some $S > 0$ such that points $y \in \mathrm{Im}(\beta) \setminus B_p(S)$ do not touch the ball $B_p(R)$ under the coarse homotopy H. We pick a $r_j > S$, and since y_j is an accumulation point in $S_p(r_j)$, there exists some $x_i^j \in S_p(r_j)$ such that $d(y_j, x_i^j) < \varepsilon$ for some $\varepsilon\mathcal{L}|R - r_j|$. Let $\varphi : [0, l_j] \to X$ be the path traced by y_j under the homotopy H. Pick a geodesic h from y_j to x_i^j. Since $len(h) = \varepsilon\mathcal{L}|R - r_j|$, this geodesic cannot touch $B_p(R)$. Also note that using g_i we can define a geodesic h' that goes from x_i to x_i^j which does not touch $B_p(R)$ either. Concatenating these three paths h', h, φ we get a path from x_i to $\mathrm{Im}(\alpha)$ which doesn't intersect $B_p(R)$, which is a contradiction.

Now we show that the path can be chosen nicely.

Since the homotopy H is coarse, in particular it is ρ-bornologous for some $\rho : \mathbb{R}_+ \to \mathbb{R}_+$. Since H is ρ-bornologous, so is φ. The maps h, h' are geodesics, so they are 1-bornologous by definition. Therefore the concatenation is ρ'-bornologous for $\rho' = \max\{\rho, 1\}$. This choice of ρ does depend on the point, only depends on our homotopy H which depends on A.

For brevity, let $x = x_i$. k_x is the length of the concatenation $h_x = \varphi \circ h \circ h'$, therefore $k_x = len(\varphi) + len(h) + len(h')$. Here len refers to the diameter of the domain of a path. So $len(\varphi) = \|y_j\| = r_j$. Since h is a geodesic of length ε, $len(h) = \varepsilon$, and h' is a geodesic between x_i

and x_i^j, so $len(h') = r_i - r_j$. Adding them,

$$k_x = r_j + \varepsilon + (r_i - r_j) = r_i + \varepsilon = \|x\| + \varepsilon < 2\,\|x\|$$

Since $\varepsilon \mathcal{L} |R - r_j|$. This concludes the proof. □

Now we can prove the previous lemma.

proof of lemma 3.5.3. Let us fix a base-point $p \in X$ and a geodesic ray $\alpha : \mathbb{R}_+ \to X$ such that $\alpha(0) = p$. For convenience, let us define a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, where $\gamma(0) = 0$, and for any $R > 0$, define $\gamma(R) = R'$ where R' is as obtained from lemma 3.5.4.

Choose an $R_1 \geq 1$. Let $R_2 = \max\{2, \gamma(R_1)\}, R_3 = \max\{3, \gamma(R_2)\}$ etc. Using lemma 3.5.4 we will construct a path $h_x : [0, k_k] \to X$ for some $k_x > 0$ such that

1. $h_x(0) = x, h_x(k_x) \in \mathrm{Im}(\alpha)$ for all $x \in U$.
2. $\mathrm{Im}(h_x) \cap B_p(R_i) = \phi$ for all $x \in U$ such that $\|x\| \geq R_{i+1}$.
3. Each h_x is ρ-bornologous, for the same ρ.
4. $k_x \leq 2\,\|x\|$.

For $i = 1$, for each $x \in U \cap B_p(R_1)$, choose any h_x satisfying (1). Property 2,3 are satisfied vacuously.

Inductively, suppose we have chosen h_x's for all $x \in U \cap B_p(R_i)$ for $i = 1, \cdots, k$ which satisfy 1,2,3,4. For each $x \in U \in A_p(R_k, R_{k+1})$, by lemma 3.5.4 there exist a path from x to $\mathrm{Im}(\alpha)$ which avoids $B_p(R_{k-1})$ (since $R_k \geq \gamma(R_{k-1})$). Let $h_x : [0, 1] \to X$ denote that path. By construction it satisfies all properties 1,2,3 and 4.

Using these paths h_x for $x \in A$, we can now construct a coarse homotopy of U which is bornologous because of 3, and proper because of 2. It will be well-defined because of 4.

□

Now we can combine lemmas 3.5.2 and 3.5.3 to get the following result:

5 Proposition. Let X be a bicombable proper geodesic metric space which is coarsely semistable at infinity, and let $\mathcal{U}$ be a dispersed family of subsets of X. Then $U = \bigcup_{A\in\mathcal{U}} A$ is a coarsely categorical set.

3.5.2 Asymptotic Dimension

Recall that a family of subsets $\mathcal{U}$ of a metric space X is called *uniformly bounded* if there is some $K > 0$ for which $diam(A) \leq K$ for all $A \in \mathcal{U}$. The family is *r-disjoint* if $d(A, A') > r$ for every $A \neq A' \in \mathcal{U}$. Here $d(A, A')$ is defined to be $\inf\{d(x, x') \mid x \in A, x' \in A'\}$. Also we recall the following definition from (Bell & Dranishnikov 2005):

3.5.3 Definition. For a subset $U \subset X$ and a family of subsets $\mathcal{V}$ of X, the *r-saturation of U* is defined as

$$N_r(U, \mathcal{V}) = U \cup \left(\bigcup V\right)$$

Where the big union inside parentheses is over all elements $V \in \mathcal{V}$ such that $d(U, V) < r$.

3.5.4 Definition. (Gromov 1993) Let X be a metric space. We say that the *asymptotic dimension* of X does not exceed n and write $\mathrm{asdim}X \leq n$ provided for every uniformly bounded open cover $\mathcal{V}$ of X there is an uniformly bounded open cover $\mathcal{U}$ of X of multiplicity $\leq n + 1$ so that $\mathcal{V}$ refines $\mathcal{U}$. We write $\mathrm{asdim}(X) = n$ if it is true that $\mathrm{asdim}(X) \leq n$ and $\mathrm{asdim}(X) \nleq n - 1$.

We are more interested in the following characterization of asymptotic dimension:

3.5.5 Theorem. Let X be a metric space. The following conditions are equivalent

1. $\mathrm{asdim}(X) \leq n$;

2. for every $r < \infty$ there exist r-disjoint families $\mathcal{U}^0, \cdots, \mathcal{U}^n$ of uniformly bounded subsets of X such that $\bigcup_i \mathcal{U}^i$ is a cover of X.

   A proof, along with other characterizations can be found in (Bell & Dranishnikov 2008).

   Now we have enough machinery to prove our main result.

3.5.6 Theorem. Let X be a bicombable proper geodesic metric space which is coarsely semistable at infinity. Then we have the following inequality

$$\text{c-cat}(X) \leq \operatorname{asdim}(X).$$

Proof. Let us fix a $\mathbb{R}_+$-base-point $\alpha : \mathbb{R}_+ \to X$ and set $p = \alpha(0) \in X$ as our base-point. Let us assume that $\operatorname{asdim}(X) \leq n$. In light of lemma 3.5.3 it is enough to construct $n+1$ families of subsets $\{\mathcal{U}^i\}_{i=0}^{i=n}$ such that

1. Each $\mathcal{U}^i$ is a dispersed family of subsets, and

2. X is covered by these $\mathcal{U}^i$'s:
$$\bigcup_{i=0}^{n} \left(\bigcup_{A \in \mathcal{U}^i} A \right) = X.$$

Now we describe the construction of these $\mathcal{U}^i$'s. We inductively define an increasing sequence of radii $\{R_j\}_{j\in\mathbb{N}}$, scales $\{\lambda_j\}_{j\in\mathbb{N}}$, and families of subsets $\mathbb{C}^i_j$ inside $A_p(R_{j-1}, R_j)$ for $j > 0$. For convenience, let $R_0 = \lambda_0 = 0$.

For the base case $j = 1$, choose a scale $\lambda_1 > 0$. Since $\operatorname{asdim}(X) \leq n$, using 3.5.5 we can get uniformly D_1-bounded, λ_1-disjoint families of subsets $\mathcal{B}^0_1, \mathcal{B}^1_1, \cdots, \mathcal{B}^n_1$ for some $D_1 > 0$ which cover X. Choose $R_1 \gg \max\{\lambda_1, D_1\}$. Now for $i = 0, 1, \cdots, n$ define

$$\mathbb{C}^i_1 = \{B \cap B_p(R_1) \mid B \in \mathcal{B}^i_1\}.$$

This concludes our base step. Now inductively assume that we have already constructed $R_j, \mathbb{C}^i_j, \lambda_j$ for $j = 1, 2, \cdots, k$. Choose a scale $\lambda_{k+1} \gg R_k$. Using 3.5.5, we get uniformly D_{k+1}-bounded λ_{k+1}-disjoint families of subsets $\mathcal{B}^0_{k+1}, \mathcal{B}^1_{k+1}, \cdots, \mathcal{B}^n_{k+1}$ which cover X, for some $D_{k+1} > 0$. We define

$$C^i_{k+1} = \{B \cap A_p(R_k, R_{k+1}) \mid B \in \mathcal{B}^i_{k+1}\}.$$

Figure 3-3 shows this construction.

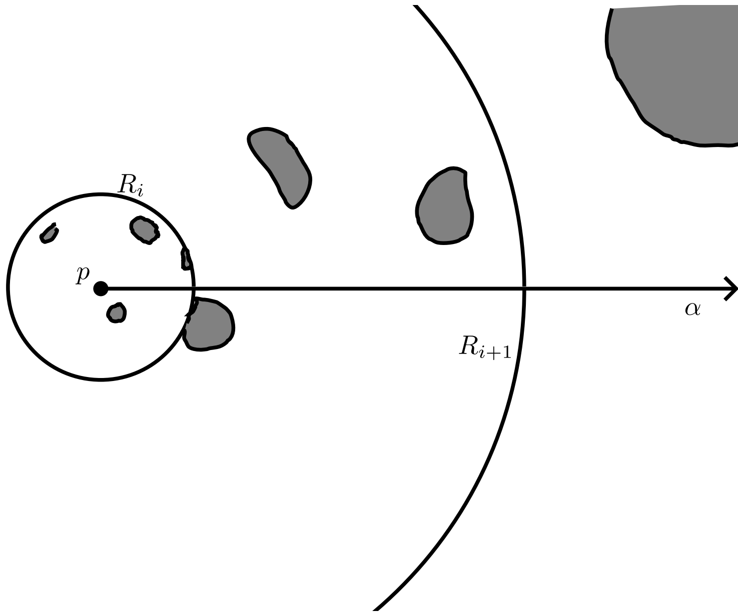

Figure 3-3. Illustration of C^i_{k+1}. The shaded blobs are the elements of C^i_{k+1}.

By our construction, $\bigcup_j \mathbb{C}^i_j$ is not quite a dispersed family, as sets in $\mathbb{C}^i_j$ can be very close to some set in $\mathbb{C}^i_{j+1}$. We need to redistribute the sets slightly to get rid of these possibilities.

For $j = 1$ define

$$\mathcal{D}^i_1 = \{C \in \mathbb{C}^i_1 \mid d(C, C') \geq \lambda_1 \text{ for all } C' \in \mathbb{C}^i_2\}.$$

For $j > 1$ define

$$D^i_j = \{N_{\lambda_{j-1}}(C, \mathbb{C}^i_{j-1}) \mid C \in \mathbb{C}^i_j,\ d(C, C') \geq \lambda_j \text{ for all } C \in \mathbb{C}^i_{j+1}\}.$$

Finally, we can define $\mathcal{U}^i = \mathcal{D}^i_1 \cup \mathcal{D}^i_2 \cup \mathcal{D}^i_3 \cup \cdots$.

- Each family $\mathcal{D}^i = \{\mathcal{D}^i_1, \mathcal{D}^i_2, \cdots\}$ consists of pairwise disjoint sets. This follows from the fact that $\lambda_{j+1} \gg \lambda_j$, so an element of $\mathbb{C}^i_j$ cannot be λ_j-close to two or more elements in $\mathbb{C}^i_{j+1}$. Boundedness follows from the fact that $R_j \gg \max\{D_j, \lambda_j\}$, so elements in $\mathbb{C}^i_j$ cannot be simultaneously λ_{j-1}-close to an element in $\mathbb{C}^i_{j-1}$ and λ_j-close to an element in $\mathbb{C}_{j+1}$.
- Each family $\mathcal{D}^i$ is a dispersed family of subsets. This follows from the fact that each set in $\mathcal{D}^i_j$ is λ_j-disjoint from others, and from sets in $\mathcal{D}^i_{j+1}$. Since element of $\mathcal{D}^i$ is bounded, we

can conclude that the function

$$\partial(r) = \inf\{d(x_1, x_2) : x_1 \neq x_2 \in U \setminus B_p(r)\}$$

goes to infinity.

Hence proved. □

The above theorem can be stated for bicombable groups now:

3.5.7 Theorem. For a bicombable group Γ that is coarsely path-connected, we have the inequality

$$\text{c-cat}(\Gamma) \leq \text{asdim}(\Gamma).$$

And as a result of claim 3.2.4, we have the following

3.5.8 Corollary. For any bicombable 1-ended discrete group Γ which is semistable at ∞ we have the inequality

$$\text{c-cat}(\Gamma) \leq \text{asdim}(\Gamma).$$

3.6 Acknowledgements

The author would like to thank his advisor Alexander Dranishnikov for all his help and insightful discussions, and Thomas Weighill for helping him understand proper vs coarse homotopy. The author would also like to thank UF Center for Applied Mathematics for summer fellowship award.

CHAPTER 3 Bibliography

Ayala, R., Domínguez Murillo, E., Márquez Pérez, A. & Quintero, A. (1992), 'Lusternik-Schnirelmann invariants in proper homotopy theory', *Pacific Journal of Mathematics* **153**(2), 201–215.

Bandelt, H.-J. & Prisner, E. (1991), 'Clique graphs and Helly graphs', *Journal of Combinatorial Theory, Series B* **51**(1), 34–45.

Bell, G. & Dranishnikov, A. (2005), 'Asymptotic dimension in Bedlewo'.

Bell, G. & Dranishnikov, A. (2008), 'Asymptotic dimension', *Topology and its Applications* **155**(12), 1265–1296.

Berstein, I. (1976), 'On the Lusternik–Schnirelmann category of Grassmannians', *Mathematical Proceedings of the Cambridge Philosophical Society* **79**(1), 129–134.

Bestvina, M., Bromberg, K. & Fujiwara, K. (2010), 'The asymptotic dimension of mapping class groups is finite'.

Bieri, R. (1981), *Homological Dimension of Discrete Groups*, Mathematics Department, Queen Mary College, University of London.

Bredon, G. E. (1997), *Sheaf Theory*, Vol. 170 of *Graduate Texts in Mathematics*, Springer, New York, NY.

Brown, K. S. (1982), *Cohomology of Groups*, Vol. 87 of *Graduate Texts in Mathematics*, Springer New York, New York, NY.

Chalopin, J., Chepoi, V., Genevois, A., Hirai, H. & Osajda, D. (2025), 'Helly groups', *Geometry & Topology* **29**(1), 1–70.

Davis, M. (2008), *The Geometry and Topology of Coxeter Groups*, London Mathematical Society Monographs Series, Princeton University Press, Princeton.

Davis, M. W. (1983), ‘Groups Generated by reflections and aspherical manifolds not covered by Euclidean space’, *Annals of Mathematics* **117**(2), 293–324.

Davis, M. W. (2001), Chapter 8 - Nonpositive Curvature and Reflection Groups, *in* ‘Handbook of Geometric Topology’, North-Holland, Amsterdam, pp. 373–422.

Dranishnikov, A. (1997), ‘On the virtual cohomological dimensions of Coxeter groups’, *Proc. Amer. Math. Soc.* **125**(7), 1885–1891.

Dranishnikov, A. (2019), ‘On dimension of product of groups’, *arXiv:1902.03201 [math]* .

Dranishnikov, A. & Kuanyshov, N. (2022), ‘On the LS-category of homomorphisms’.

Dranishnikov, A. N. (2015), ‘The LS category of the product of lens spaces’, *Algebr. Geom. Topol.* **15**(5), 2985–3010.

Dranishnikov, A. N. & Rudyak, Y. B. (2009), ‘On the Berstein–Svarc theorem in dimension 2’, *Mathematical Proceedings of the Cambridge Philosophical Society* **146**(2), 407–413.

Dranishnikov, A. & Smith, J. (2006), ‘Asymptotic dimension of discrete groups’, *Fundamenta Mathematicae* **189**(1), 27–34.

Eilenberg, S. & Ganea, T. (1957), ‘On the Lusternik-Schnirelmann Category of Abstract Groups’, *Annals of Mathematics* **65**(3), 517–518.

Engel, A. & Wulff, C. (2021), ‘Coronas for properly combable spaces’.

Fischer, H. (2003), ‘Boundaries of right-angled Coxeter groups with manifold nerves’, *Topology* **42**(2), 423–446.

Grant, M. (2022), ‘Cohomological dimension of a homomorphism’.

Gromov, M. (1987), Hyperbolic Groups, *in* ‘Essays in Group Theory’, Springer, New York, NY, pp. 75–263.

Gromov, M. (1993), *Asymptotic Invariants of Infinite Groups. Geometric Group Theory: Volume 2*, Vol. 2, Cambridge University Press.

Gromov, M. (1996), Positive Curvature, Macroscopic Dimension, Spectral Gaps and Higher Signatures, *in* 'Functional Analysis on the Eve of the 21st Century Volume II: In Honor of the Eightieth Birthday of I. M. Gelfand', Birkhäuser, Boston, MA, pp. 1–213.

Hatcher, A. (2002), *Algebraic Topology*, Cambridge University Press, Cambridge ; New York.

Hosaka, T. & Yokoi, K. (n.d.), 'THE BOUNDARY AND THE VIRTUAL COHOMOLOGICAL DIMENSION OF COXETER GROUPS'.

Ji, L. (2004), 'Asymptotic dimension and the integral K-theoretic Novikov conjecture for arithmetic groups', *Journal of Differential Geometry* **68**(3), 535–544.

Jr, D. R. M. (1961), 'Cartesian products of contractible open manifolds', *Bulletin of the American Mathematical Society* **67**(5), 510–514.

Kato, T. (2000), 'Asymptotic Lipschitz maps, combable groups and higher signatures', *GAFA, Geom. funct. anal.* **10**(1), 51–110.

Kropholler, P., Linnell, P. & Lück, W. (2009), Groups of small homological dimension and the Atiyah conjecture, *in* 'Geometric and Cohomological Methods in Group Theory', London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, pp. 272–277.

Margolis, A. (2024), 'Coarse homological invariants of metric spaces'.

McMillan, D. R. & Zeeman, E. C. (1962), 'On Contractible Open Manifolds', *Mathematical Proceedings of the Cambridge Philosophical Society* **58**(2), 221–224.

Mitchener, P. D., Norouzizadeh, B. & Schick, T. (2020), 'Coarse Homotopy Groups', *Mathematische Nachrichten* **293**(8), 1515–1533.

Roe, J. (2003), ‘Lectures on Coarse Geometry’, https://www.ams.org/ulect/031.

Roe, J. (2005), ‘Hyperbolic groups have finite asymptotic dimension’, *Proc. Amer. Math. Soc.* **133**(9), 2489–2490.

Scott, J. (2022), ‘On the topological complexity of maps’, *Topology and its Applications* **314**, 108094.

Srinivasan, T. (2015), The Lusternik-Schirelmann Category of Peano Cintinua, PhD thesis, University of Florida.

Stallings, J. (1962), ‘The Piecewise-Linear Structure of Euclidean Space’, *Mathematical Proceedings of the Cambridge Philosophical Society* **58**(3), 481–488.

Švarc, A. S. (1965), ‘The genus of a fiber space’, *American Mathematical Society Translations: Series 2* pp. 151–155.

Weighill, T. (2019), ‘Lifting coarse homotopies’.

Yu, G. (1998), ‘The Novikov Conjecture for Groups with Finite Asymptotic Dimension’, *Annals of Mathematics* **147**(2), 325–355.